Quantum Computation and Error Correction **Fall 2026**

Sheet 4 — Solutions

TA: Viktoriia Pinchenkova, viktoriia.pinchenkova@unibas.ch, office 4.6

To be discussed on Tuesday, October 13, 16:00

Submission deadline: Monday, October 12, 23:59 (Basel time)

Please write your full name and student ID number directly on the assignment. **We accept only handwritten solutions.**

Conventions. All vector spaces are finite-dimensional over $\mathbb{C}$. For two qubits, use the ordered basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$, where $|xy\rangle = |x\rangle \otimes |y\rangle$ and the first label refers to the first qubit. The single-qubit matrices used below are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

1. Spectral Projectors and Pauli Rotations

Let O be a Hermitian operator whose only eigenvalues are $+1$ and -1.

(a) Derive the spectral projectors $P_\pm = (I \pm O)/2$. Verify that they are orthogonal, i.e.,

$$P_\pm^2 = P_\pm, \qquad P_\pm^\dagger = P_\pm.$$

(b) Write six spectral projectors in ket–bra form of X, Y and Z.

(c) Let the pure qubit state $|\psi\rangle$ correspond to the Bloch vector $\vec{r}$. Determine the Bloch vectors corresponding to $X|\psi\rangle$, $Y|\psi\rangle$, and $Z|\psi\rangle$. Give a geometric interpretation of the action of each Pauli matrix on the Bloch sphere.

Solution

(a) The spectral decomposition of the Hermitian operator O and the completeness relation give

$$O = P_{+1} - P_{-1}, \qquad I = P_{+1} + P_{-1}.$$

Adding them gives $I + O = 2P_{+1}$ and subtracting gives $I - O = 2P_{-1}$, that is,

$$P_{\pm 1} = \frac{I \pm O}{2}.$$

We have $O^2 = I$ (each eigenvalue squares to 1). Consequently,

$$P_\pm^2 = \frac{I \pm 2O + O^2}{4} = \frac{I \pm O}{2} = P_\pm,$$
$$P_\pm^\dagger = \frac{I \pm O^\dagger}{2} = P_\pm.$$

(b) For X, the eigenvalue equation gives $v_2 = \pm v_1$, so its unit eigenvectors are $|\pm\rangle$. For Y, the first row gives $-\mathrm{i}v_2 = \pm v_1$, so

$$|y_+\rangle = \frac{|0\rangle + \mathrm{i}|1\rangle}{\sqrt{2}}, \qquad |y_-\rangle = \frac{|0\rangle - \mathrm{i}|1\rangle}{\sqrt{2}}.$$

The eigenvectors of Z are the computational basis vectors. The projectors are thus

Operator	$+1$ eigenvector	-1 eigenvector	P_+	P_-
X	$\lvert+\rangle$	$\lvert-\rangle$	$\lvert+\rangle\langle+\rvert$	$\lvert-\rangle\langle-\rvert$
Y	$\lvert y_+\rangle$	$\lvert y_-\rangle$	$\lvert y_+\rangle\langle y_+\rvert$	$\lvert y_-\rangle\langle y_-\rvert$
Z	$\lvert0\rangle$	$\lvert1\rangle$	$\lvert0\rangle\langle0\rvert$	$\lvert1\rangle\langle1\rvert$

(c) Write

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}\,|1\rangle\,, \qquad \vec{r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta),$$

and recall that global phases may be discarded.

Action of Z. $Z\,|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle - \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}\,|1\rangle = \cos\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{\mathrm{i}(\phi+\pi)}\sin\frac{\theta}{2}\,|1\rangle$, so $(\theta,\phi)\mapsto(\theta,\phi+\pi)$ and $\vec{r}\mapsto(-r_x,-r_y,r_z)$.

Action of X. $X\,|\psi\rangle = \mathrm{e}^{\mathrm{i}\phi}\big(\sin\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{-\mathrm{i}\phi}\cos\frac{\theta}{2}\,|1\rangle\big)$; dropping the global phase $\mathrm{e}^{\mathrm{i}\phi}$ gives $(\theta,\phi)\mapsto(\pi-\theta,-\phi)$ and $\vec{r}\mapsto(r_x,-r_y,-r_z)$.

Action of Y. Using $Y\,|0\rangle = \mathrm{i}\,|1\rangle$, $Y\,|1\rangle = -\mathrm{i}\,|0\rangle$, $Y\,|\psi\rangle = -\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\big(\sin\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{\mathrm{i}(\pi-\phi)}\cos\frac{\theta}{2}\,|1\rangle\big)$; dropping the global phase gives $(\theta,\phi)\mapsto(\pi-\theta,\pi-\phi)$ and $\vec{r}\mapsto(-r_x,r_y,-r_z)$.

In each case one coordinate is preserved and the other two change sign: this is a rotation by π (a half-turn) about the corresponding coordinate axis. So X, Y, Z act on the Bloch sphere as π-rotations about the x-, y- and z-axis respectively; the fixed points are the eigenvectors ($|\pm\rangle$ for X, $|\pm\mathrm{i}\rangle = (|0\rangle \pm \mathrm{i}\,|1\rangle)/\sqrt{2}$ for Y, and $|0\rangle, |1\rangle$ for Z), and the fact that two half-turns about the same axis give the identity is the geometric content of $\sigma_j^2 = I$. Combining two different half-turns gives a half-turn about the third axis up to phase, which is the geometric content of $XY = \mathrm{i}Z$.

---

2. Spectral Decomposition and a Function of an Operator

In the computational basis $(|0\rangle, |1\rangle)$, consider

$$A = \begin{pmatrix} 2 & 1-\mathrm{i} \\ 1+\mathrm{i} & 3 \end{pmatrix}.$$

(a) Verify that A is Hermitian. Find its eigenvalues $\lambda_1 < \lambda_2$ and corresponding normalised eigenvectors $|v_1\rangle, |v_2\rangle$. Check that these vectors form an orthonormal basis.

To construct the change-of-basis matrix U, place these eigenvectors, expressed in the computational basis, in its columns:

$$U = \big(|v_1\rangle\ \ |v_2\rangle\big).$$

Construct $D = \mathrm{diag}(\lambda_1, \lambda_2)$, keeping the eigenvalues in the same order as the columns of U. Verify that $U^\dagger U = I_2$ and $U^\dagger A U = D$.

Meaning of the change of basis. If x and c are the coordinate columns of the same vector in the computational basis and the eigenbasis, respectively, then

$$x = Uc, \qquad c = U^\dagger x,$$

where U is unitary. Thus D represents the same operator as A, but in its eigenbasis, where each coordinate is multiplied by the corresponding eigenvalue.

(b) Construct the spectral projectors P_λ as outer products and as matrices. Write the spectral decomposition $A = \sum_\lambda \lambda P_\lambda$. Find $\operatorname{tr} A$ from the eigenvalues and explain why this works.

(c) Is A positive definite? Use its spectral decomposition to find the positive square root $B = \sqrt{A}$, both in projector form and as a matrix.

Solution

(a) The diagonal entries are real and the off-diagonal entries are complex conjugates, so $A^\dagger = A$. Hence A admits an orthonormal eigenbasis. Its characteristic polynomial is

$$\det(A - \lambda I_2) = (2-\lambda)(3-\lambda) - (1-\mathrm{i})(1+\mathrm{i}) = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4).$$

Thus $\lambda_1 = 1$ and $\lambda_2 = 4$.

Writing a trial eigenvector as $(z_1, z_2)^{\mathrm{T}}$, for $\lambda = 1$ the first row of the eigenvalue equation gives $z_1 = -(1-\mathrm{i})z_2$. For $\lambda = 4$, the second row gives $z_2 = (1+\mathrm{i})z_1$. Normalised eigenvectors are therefore

$$|v_1\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} -1+\mathrm{i} \\ 1 \end{pmatrix}, \qquad |v_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1+\mathrm{i} \end{pmatrix}.$$

Both have squared norm 1, and

$$\langle v_1 | v_2 \rangle = \frac{(-1-\mathrm{i}) + (1+\mathrm{i})}{3} = 0.$$

They therefore form an orthonormal basis.

Putting these vectors in columns and the corresponding eigenvalues on the diagonal gives

$$U = \frac{1}{\sqrt{3}}\begin{pmatrix} -1+\mathrm{i} & 1 \\ 1 & 1+\mathrm{i} \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

The orthonormal columns imply $U^\dagger U = I_2$. Applying A to each column of U multiplies that column by its eigenvalue, so $AU = UD$. Consequently,

$$U^\dagger A U = D, \qquad A = U D U^\dagger.$$

(b) The spectral projectors are

$$P_1 = |v_1\rangle\langle v_1| = \frac{1}{3}\begin{pmatrix} 2 & -1+\mathrm{i} \\ -1-\mathrm{i} & 1 \end{pmatrix}, \qquad P_4 = |v_2\rangle\langle v_2| = \frac{1}{3}\begin{pmatrix} 1 & 1-\mathrm{i} \\ 1+\mathrm{i} & 2 \end{pmatrix}.$$

The spectral decomposition is

$$A = P_1 + 4P_4 = |v_1\rangle\langle v_1| + 4\,|v_2\rangle\langle v_2|\,.$$

The trace is invariant under any change of basis. Indeed, for any invertible change-of-basis matrix S, cyclicity of the trace gives

$$\operatorname{tr}(S^{-1}AS) = \operatorname{tr}(ASS^{-1}) = \operatorname{tr} A.$$

In an eigenbasis, the matrix is diagonal and its diagonal entries are the eigenvalues. Therefore,

$$\operatorname{tr} A = \operatorname{tr}(U^\dagger A U) = \operatorname{tr} D = \lambda_1 + \lambda_2 = 1 + 4 = 5,$$

in agreement with the original diagonal entries, $2 + 3 = 5$.

(c) Start from the spectral decomposition:

$$A = |v_1\rangle\langle v_1| + 4\,|v_2\rangle\langle v_2|\,.$$

For any nonzero $|w\rangle$, this gives

$$\begin{aligned}\langle w|A|w\rangle &= \langle w|v_1\rangle\,\langle v_1|w\rangle + 4\,\langle w|v_2\rangle\,\langle v_2|w\rangle \\ &= |\,\langle v_1|w\rangle\,|^2 + 4|\,\langle v_2|w\rangle\,|^2 > 0.\end{aligned}$$

The two overlaps cannot both vanish because $|v_1\rangle\,, |v_2\rangle$ form a basis and $|w\rangle \neq 0$. Thus A is positive definite, consistently with both of its eigenvalues being strictly positive.

Apply the positive scalar square root to each eigenvalue:

$$\boxed{B = \sqrt{A} = P_1 + 2P_4 = \frac{1}{3}\begin{pmatrix} 4 & 1-\mathrm{i} \\ 1+\mathrm{i} & 5 \end{pmatrix}.}$$

Using the projector identities from part (b),

$$\begin{aligned}B^2 &= (P_1 + 2P_4)^2 \\ &= P_1^2 + 2P_1P_4 + 2P_4P_1 + 4P_4^2 \\ &= P_1 + 4P_4 = A.\end{aligned}$$

Moreover, B is Hermitian and has eigenvalues 1 and 2, so it is itself positive definite. This illustrates the spectral rule

$$f(A) = \sum_{\lambda} f(\lambda) P_\lambda.$$

---

3. What the Spectral Theorem Does and Does Not Say

Terminology. A matrix M is *diagonalisable by similarity* if $S^{-1}MS$ is diagonal for some invertible S. It is *unitarily diagonalisable* if $U^\dagger MU$ is diagonal for some unitary U. The columns of S form an eigenbasis; for a unitary U, that eigenbasis must also be orthonormal.

Consider

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

(a) Compute $M^\dagger M$ and $MM^\dagger$. Is M normal? Find the eigenvalues and eigenspaces of M. Is it diagonalisable by similarity?

(b) Find the eigenvalues and eigenvectors of N, and an invertible S for which $S^{-1}NS$ is diagonal. Are its two eigenspaces orthogonal? Check whether N is normal by comparing $N^\dagger N$ and $NN^\dagger$.

(c) State the spectral theorem for complex matrices and use it to decide whether either matrix is unitarily diagonalisable. Explain why the result for N in part (b) does not contradict the theorem. Also check $N^2 = I_2$: does this identity make N unitary?

Solution

(a) Taking the conjugate transpose and multiplying gives

$$M^\dagger = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M^\dagger M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad MM^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

These products differ, so M is not normal. Since $\det(M - \lambda I_2) = (1-\lambda)^2$, its only eigenvalue is 1, with algebraic multiplicity two. The eigenvalue equation is

$$(M - I_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} = \mathbf{0},$$

so the eigenspace is $\mathrm{span}\{(0,1)^{\mathrm{T}}\}$. It is one-dimensional. Two independent eigenvectors would be required to form a basis of $\mathbb{C}^2$, so M is not diagonalisable even by a general similarity transformation.

(b) The characteristic polynomial is

$$\det(N - \lambda I_2) = (1-\lambda)(-1-\lambda) = \lambda^2 - 1.$$

Thus $\lambda = \pm 1$. The respective equations are $v_1 - 2v_2 = 0$ and $v_1 = 0$, so we may take

$$|v_+\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad |v_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These eigenvectors are independent. With

$$S = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix},$$

the eigenvalue equations give $S^{-1}NS = \mathrm{diag}(1,-1)$. However, $\langle v_+|v_-\rangle = 1 \neq 0$. Even after normalisation the overlap is $1/\sqrt{5}$, so rescaling the two vectors cannot make this eigenbasis orthogonal. Each eigenspace is one-dimensional, leaving no alternative eigenvector directions to choose.

Directly,

$$N^\dagger N = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad NN^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

so N is not normal.

(c) The finite-dimensional complex spectral theorem states that a matrix is unitarily diagonalisable if and only if it is normal. Neither M nor N is normal, so neither admits an orthonormal eigenbasis. The matrix N does admit a non-orthogonal eigenbasis; ordinary similarity diagonalisation does not require normality. For comparison, the Hermitian matrix A in Problem 2 satisfies both notions of diagonalisation:

Matrix	Normal	Similarity diagonalisation	Unitary diagonalisation
A (Problem 2)	yes	yes	yes
M	no	no	no
N	no	yes	no

Although direct multiplication gives $N^2 = I_2$, part (b) shows $N^\dagger N \neq I_2$. Hence N is not unitary. The identity $N^2 = I_2$ only establishes $N^{-1} = N$; unitarity additionally requires $N^{-1} = N^\dagger$.

4. Tensor Products, CNOT and Entanglement

Background: product states and entanglement.

All states considered in this exercise are *pure states*, described by normalised state vectors. A general pure state of two qubits is

$$|\psi\rangle = c_{00}\,|00\rangle + c_{01}\,|01\rangle + c_{10}\,|10\rangle + c_{11}\,|11\rangle\,, \qquad \sum_{x,y=0}^{1} |c_{xy}|^2 = 1,$$

where $|xy\rangle = |x\rangle \otimes |y\rangle$.

The state is called a *product state* if it can be written as

$$|\psi\rangle = |a\rangle \otimes |b\rangle\,,$$

where $|a\rangle$ and $|b\rangle$ are normalised single-qubit states. Thus the joint state can be specified by assigning a pure state to each qubit separately.

Writing

$$|a\rangle = a_0\,|0\rangle + a_1\,|1\rangle\,, \qquad |b\rangle = b_0\,|0\rangle + b_1\,|1\rangle\,,$$

and expanding their tensor product gives

$$|a\rangle \otimes |b\rangle = a_0 b_0\,|00\rangle + a_0 b_1\,|01\rangle + a_1 b_0\,|10\rangle + a_1 b_1\,|11\rangle\,.$$

Therefore, to test whether a state is a product state, try to find single-qubit coefficients satisfying

$$c_{00} = a_0 b_0, \qquad c_{01} = a_0 b_1, \qquad c_{10} = a_1 b_0, \qquad c_{11} = a_1 b_1.$$

If no such factorisation exists, the pure state is called *entangled*. To establish entanglement, show that the coefficient equations cannot all be satisfied.

A superposition of several computational-basis vectors can still be a product state. For example,

$$\frac{|00\rangle + |01\rangle}{\sqrt{2}} = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

Here the first qubit is in $|0\rangle$ and the second is in $(|0\rangle + |1\rangle)/\sqrt{2}$, so the state is not entangled. The relevant question is whether the joint state factorises, rather than how many terms appear in its basis expansion.

(a) Compute the 4×4 matrices $X \otimes I_2$ and $I_2 \otimes X$ and describe which qubit each flips. Show, using the mixed-product rule, that these different operators commute.

(b) The controlled-NOT gate, with the first qubit as control and the second as target, is

$$C = \mathrm{CNOT} = |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes X.$$

Determine C's action on all four computational basis vectors and write its matrix.

(c) Decide which of the following normalised pure states are product states and which are entangled. Factor any product state:

$$|\chi\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}, \qquad \left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

(d) Compute $C(H \otimes I_2)\,|00\rangle$. Show that an operator of the form $A \otimes B$ maps a product vector to another product vector, whenever the output is nonzero. Use this to explain why C cannot be written as $A \otimes B$.

Solution

(a) The Kronecker products are

$$X \otimes I_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad I_2 \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The first flips the first qubit, $|xy\rangle \mapsto |1-x, y\rangle$; the second flips the second qubit, $|xy\rangle \mapsto |x, 1-y\rangle$. For example, their actions on $|00\rangle$ give $|10\rangle$ and $|01\rangle$ respectively, so they are different operators. Nevertheless,

$$(X \otimes I_2)(I_2 \otimes X) = (XI_2) \otimes (I_2X) = X \otimes X,$$
$$(I_2 \otimes X)(X \otimes I_2) = (I_2X) \otimes (XI_2) = X \otimes X.$$

Swapping tensor factors changes which subsystem is acted on. Composing operations on different subsystems in either order gives the same result.

(b) The first term of C acts when the control is 0, while the second acts when the control is 1. Hence

$$|00\rangle \mapsto |00\rangle\,, \quad |01\rangle \mapsto |01\rangle\,, \quad |10\rangle \mapsto |11\rangle\,, \quad |11\rangle \mapsto |10\rangle\,.$$

These images form the columns of

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(c) The first state factors as

$$|\chi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |+\rangle \otimes |-\rangle\,.$$

For the second state, suppose a factorisation existed:

$$\left|\Phi^+\right\rangle = (a_0\,|0\rangle + a_1\,|1\rangle) \otimes (b_0\,|0\rangle + b_1\,|1\rangle).$$

Matching coefficients would require

$$a_0b_0 = a_1b_1 = \frac{1}{\sqrt{2}}, \qquad a_0b_1 = a_1b_0 = 0.$$

The first pair forces all four factors a_0, a_1, b_0, b_1 to be nonzero, contradicting the second pair. Thus $|\Phi^+\rangle$ is entangled.

(d) The rightmost operator acts first:

$$(H \otimes I_2)\,|00\rangle = |+\rangle \otimes |0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}},$$

and then

$$C(H \otimes I_2)\,|00\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \left|\Phi^+\right\rangle .$$

On the other hand, by the defining action of tensor-product operators,

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = (A\,|v\rangle) \otimes (B\,|w\rangle).$$

If this vector is nonzero, normalising it preserves its product form: the norm of a tensor product is the product of the norms of its factors. If it is zero, it does not represent a quantum state. Since C takes the product input $|+\rangle \otimes |0\rangle$ to the entangled state $|\Phi^+\rangle$, it cannot factor as $A \otimes B$.

---

5. Hadamards on an n-Qubit Register

Let $n \geq 1$ and label the computational basis by $i, j \in \{0, \ldots, 2^n - 1\}$. Use the binary convention

$$i = \sum_{k=1}^{n} i_k 2^{n-k}, \qquad |i\rangle = |i_1\rangle \otimes \cdots \otimes |i_n\rangle\,, \qquad i_k \in \{0, 1\},$$

and similarly for j.

(a) Show that

$$\left\langle i\middle|H^{\otimes n}\middle|j\right\rangle = \frac{(-1)^{i\cdot j}}{\sqrt{2^n}}, \qquad i \cdot j = \sum_{k=1}^{n} i_k j_k \pmod 2.$$

Hint: First express the four single-qubit matrix entries $\langle a|H|b\rangle$ in one formula, with $a, b \in \{0, 1\}$.

(b) Using the expressions form (a), find $H^{\otimes n}\,|0\rangle^{\otimes n}$. How many computational-basis amplitudes are nonzero, and what are the corresponding measurement probabilities? Does this operation create entanglement from $|0\rangle^{\otimes n}$? Justify your answer by writing the output as a tensor product.

Solution

(a) The single-qubit Hadamard has entries

$$\langle a|H|b\rangle = \frac{(-1)^{ab}}{\sqrt{2}}.$$

The sign is negative only when $a = b = 1$, exactly as in the matrix of H. Tensor-product matrix elements factorise, so

$$\begin{aligned}\left\langle i\middle|H^{\otimes n}\middle|j\right\rangle &= \prod_{k=1}^{n} \langle i_k|H|j_k\rangle = \prod_{k=1}^{n} \frac{(-1)^{i_k j_k}}{\sqrt{2}} \\ &= \frac{(-1)^{\sum_{k=1}^{n} i_k j_k}}{2^{n/2}} = \frac{(-1)^{i\cdot j}}{\sqrt{2^n}}.\end{aligned}$$

Only the parity of the exponent matters, so it may be reduced modulo 2.

(b) We denote $H^{\otimes n}\,|0\rangle^{\otimes n} = |\Psi\rangle$ and write it in the computentional basis

$$H^{\otimes n}\,|0\rangle^{\otimes n} = |\Psi\rangle = \sum_{i=0}^{2^n-1} \langle i|\,|\Psi\rangle\,|i\rangle\,.$$

$$\langle i|\,|\Psi\rangle = \langle i|\,H^{\otimes n}\,|0\rangle^{\otimes n} = \frac{(-1)^{i\cdot 0}}{\sqrt{2^n}}.$$

For $|j\rangle = |0\rangle^{\otimes n}$, every bit j_k is zero. Thus all 2^n amplitudes are $1/\sqrt{2^n}$:

$$H^{\otimes n}\,|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle\,.$$

A computational-basis measurement returns each bit string with probability 2^{-n}; the probabilities sum to 1. The same state is

$$(H\,|0\rangle)^{\otimes n} = |+\rangle^{\otimes n} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^{\otimes n}.$$

It is a product state, so the Hadamard layer creates no entanglement from this input. Having many nonzero computational-basis amplitudes alone does not imply entanglement.