QECC and Statistical Mechanics

We have found that the planar code can be used to correct errors

This requires a decoding algorithm to look at the syndrome (anyon configuration) and determine the best way to return the code to the stabilizer space

Ideally, the combination of error and correction will act as identity on the stored logical qubit

There exists a threshold noise strength for good decoders, below which the probability of failure decays with L

This threshold depends on both the code and decoder. But what is the maximum possible threshold for any given code?

This can be determined through a connection between error correction and statistical mechanics

We will look at this for the case of perfect measurements

An ideal decoder for the planar code would determine the probabilities for each equivalence class of errors, and correct according to the most likely

Today we'll use the flux syndrome, and label the classes

even # σ_x's on lhs $\therefore$ class $\mathbb{1}$

odd # σ_x's on lhs $\therefore$ class X

The probability of each class is then

$$P(E \in \mathbb{1} | S) = \sum_{E \in \mathbb{1} \cap S} p^{N_E} p_0^{n - N_E} \qquad P(E \in X | S) = \sum_{E \in X \cap S} p^{N_E} p_0^{n - N_E}$$

where - N_E is the # σ_x errors in E

- $E \in \mathbb{1} \cap S$ means all E consistent with S and in class $\mathbb{1}$, etc

Calculating these probabilties and applying an E_c from the most likely class then gives logical error rate

$$P(s) = \min\Big(P(E \in \mathbb{1} | S), P(E \in X | S)\Big)$$

Let's use

$$p^{N_E} p_0^{n-N_E} = p_0^n \left(\frac{p}{p_0}\right)^{N_E}$$

And consider the partition functions

$$Z(E \in \mathbb{1} | S) = \frac{P(E \in \mathbb{1} | S)}{p_0^n} = \sum_{E \in \mathbb{1} \cap S} \left(\frac{p}{p_0}\right)^{N_E}$$

$$Z(E \in X | S) = \frac{P(E \in X | S)}{p_0^n} = \sum_{E \in X \cap S} \left(\frac{p}{p_0}\right)^{N_E}$$

The ratio of the Z's is the same as that for the P's, so we can also use these to find out which class is bigger

Sometimes different models have the same partition function, allowing results to be reused

Can we find another model with this partition function?

Yes! And a well known one too! The RBIM

Planar code with σ_x errors $\equiv$ Random bond Ising model in 2D

The Ising model is a toy model of magnetism

Variants of the Ising model are equivalent to many interesting models

It is well known and well studied, so many results have already been found

Once we map the planar code to the Ising model, we can easily determine if there is a threshold, and its value

The Ising Model

Static spins with nearest neighbour interactions

$$H = -\sum_{\langle ij \rangle} J_{ij}\, \sigma_z^i \sigma_z^j$$

For $J_{ij} > 0$

$$\varepsilon_{ij} = |J_{ij}| \quad \text{for} \quad |01\rangle, \; |10\rangle$$

$$\varepsilon_{ij} = -|J_{ij}| \quad \text{for} \quad |00\rangle, \; |11\rangle$$

Ferromagnetic bond: wants spins to align

For $J_{ij} < 0$

$$\varepsilon_{ij} = |J_{ij}| \quad \text{for} \quad |00\rangle, \; |00\rangle$$

$$\varepsilon_{ij} = -|J_{ij}| \quad \text{for} \quad |01\rangle, \; |10\rangle$$

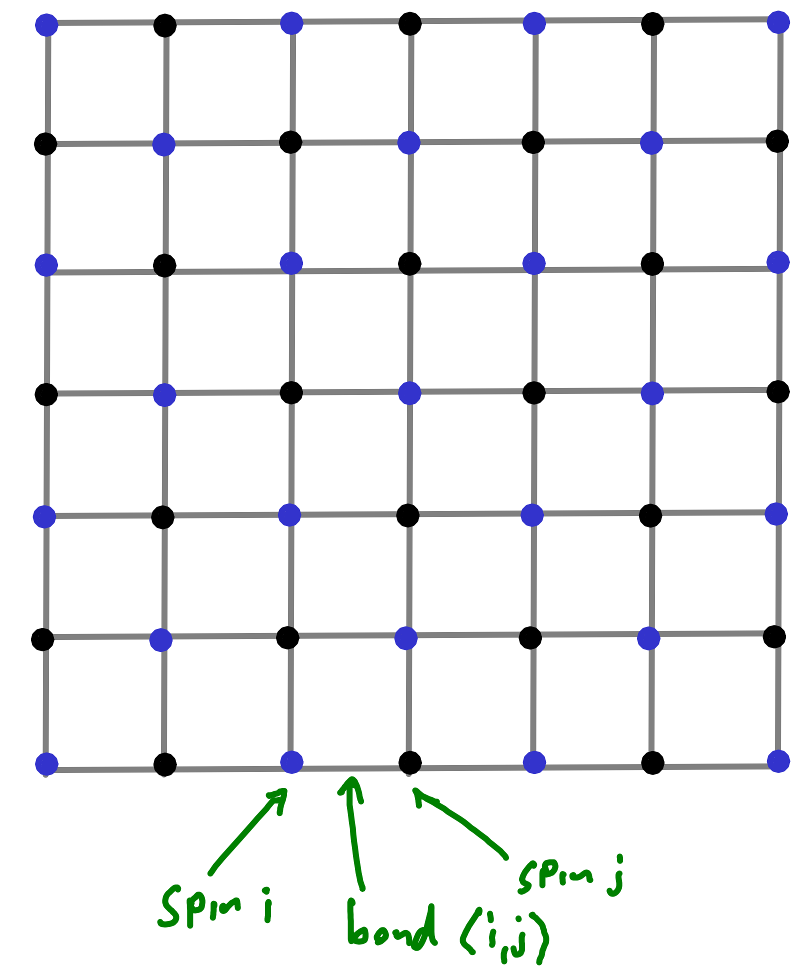
Anti-ferromagnetic bond: wants spins to anti-align

We consider the Ising model on a 2D square lattice

If the couplings are uniform: $J_{ij} = J \quad \forall i,j$

It is a ferromagnet for $J > 0$

and an antiferromagnet for $J < 0$



Two Hamiltonians are isospectral if

$$UHU^{\dagger} = H' \qquad U \text{ is unitary}$$

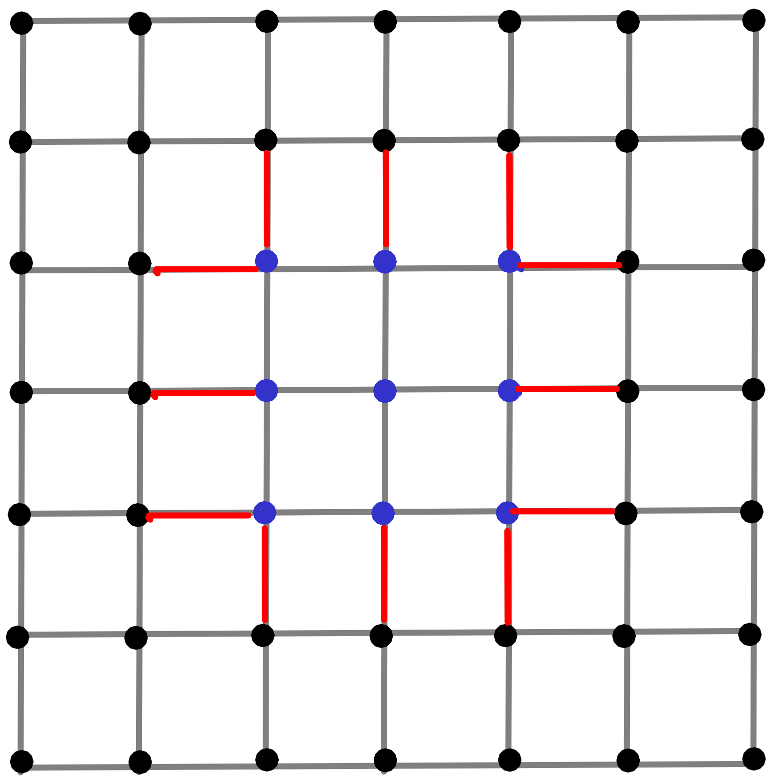
This means that they are equivalent, differing only by a rotation

If we define $\quad U = \prod_{j \in \text{blue}} \sigma_x^j$

Then $\quad U H_{\text{ferro}} U^{\dagger} = H_{\text{antiferro}}$

So these cases are equivalent

Now uniform magnitude $|J_{ij}| = J > 0$ but not uniform sign



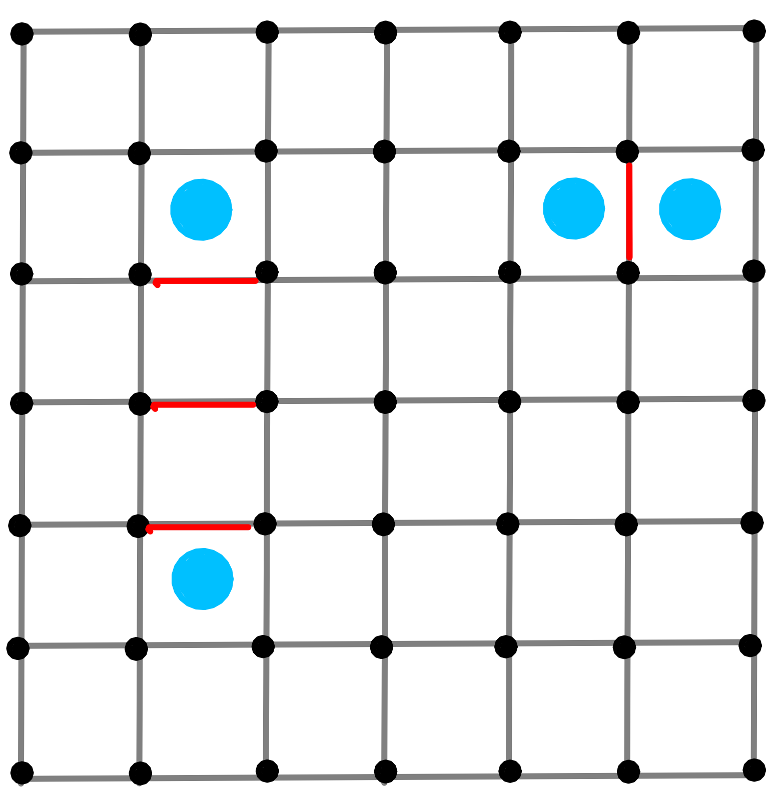
Example: $J_{ij} = -J$ on red bonds only

If the -ve bonds form loops (even number round each plaquette) we can find a U s.t.

$$U H_{loops} U^{\dagger} = H_{ferro}$$

The blue spins live inside the loops

So also equivalent to FM case

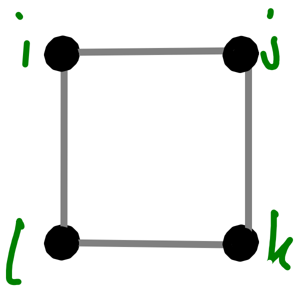


For open strings, no such U exists. Hamiltonian is not equivalent

Different spectrum

Different properties

This is due to 'frustrated plaquettes' around which all bonds cannot be satisfied



All terms around a plaquette can be simulanteously in their own gs if $J_{ij} J_{jk} J_{kl} J_{li} > 0$

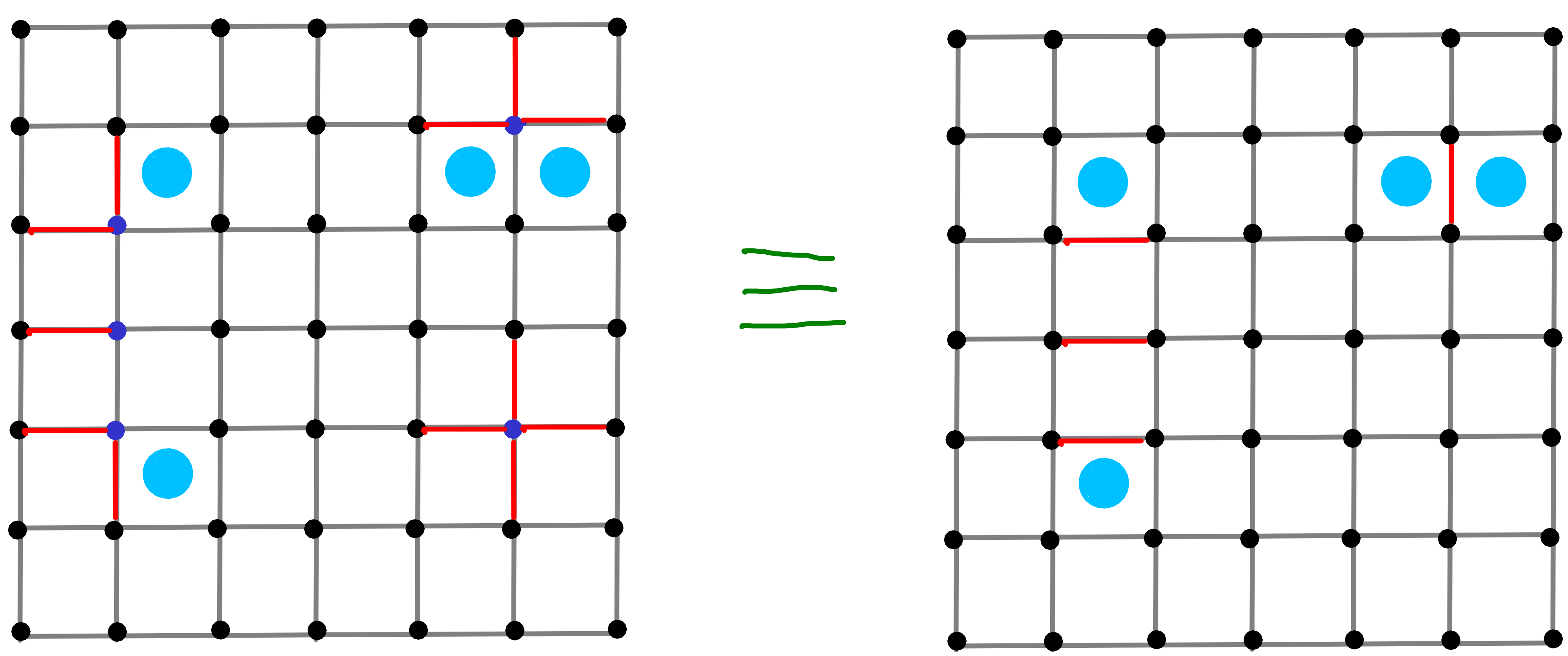
but not if $J_{ij} J_{jk} J_{kl} J_{li} < 0$

The latter are the frustrated plaquettes

odd number of -ve couplings

endpoints of strings of -ve couplings

Two Hamiltonians are unitarily equivalent iff they share the same set of frustrated plaquettes



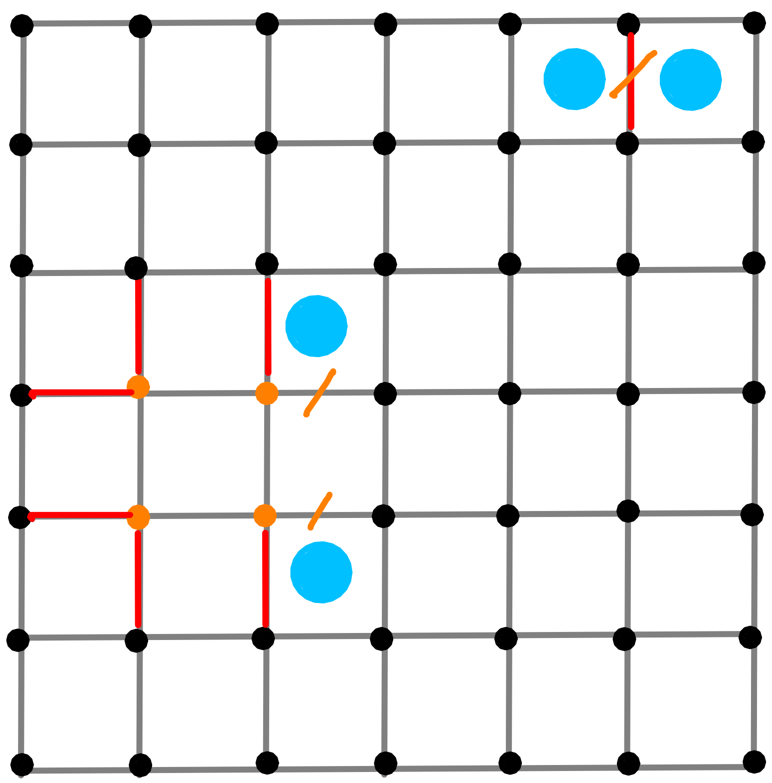
Blue spins live inside the loops formed when the two sets of AFM bonds are combined

Random Bond Ising Model

The RBIM model has uniform coupling but assigns signs randomly $P(J_{ij} = -J) = q$ $P(J_{ij} = +J) = 1-q$

For small q, what is the gs (minimize # unsatisfied bonds)?

It is a MWPM of the frustrated plaquettes



Here black spins are 0 and orange are 1 (or vice-versa)

Unsatisfied bonds have /

Most spins are aligned along a given direction, with only a minority in the opposite direction

Pretty much an FM gs

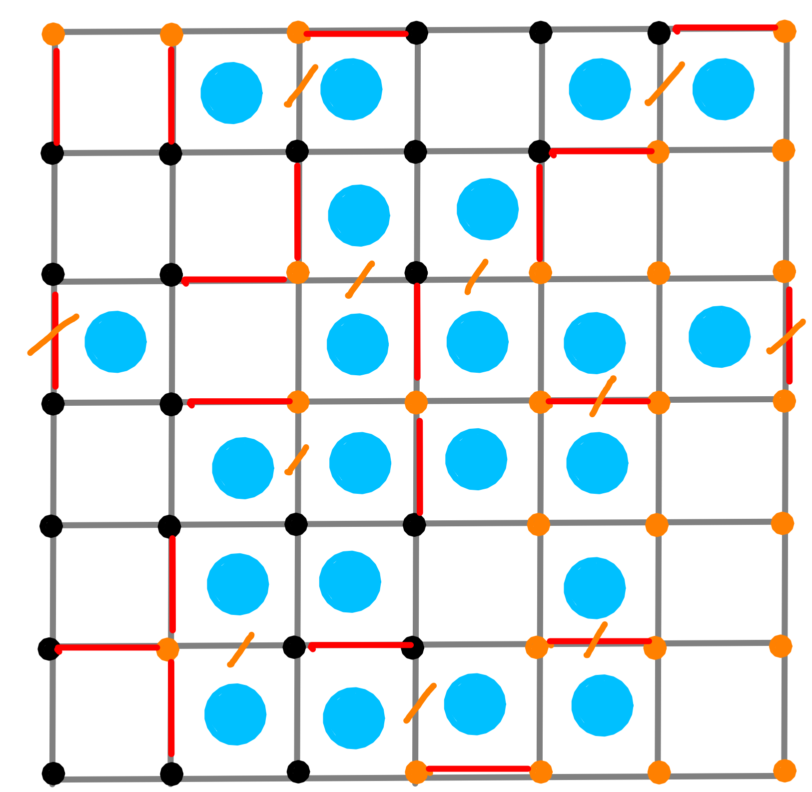
For higher q, such as $q \approx \frac{1}{4}$, gs has

Lots of frustration

Lots of anti-aligned spins

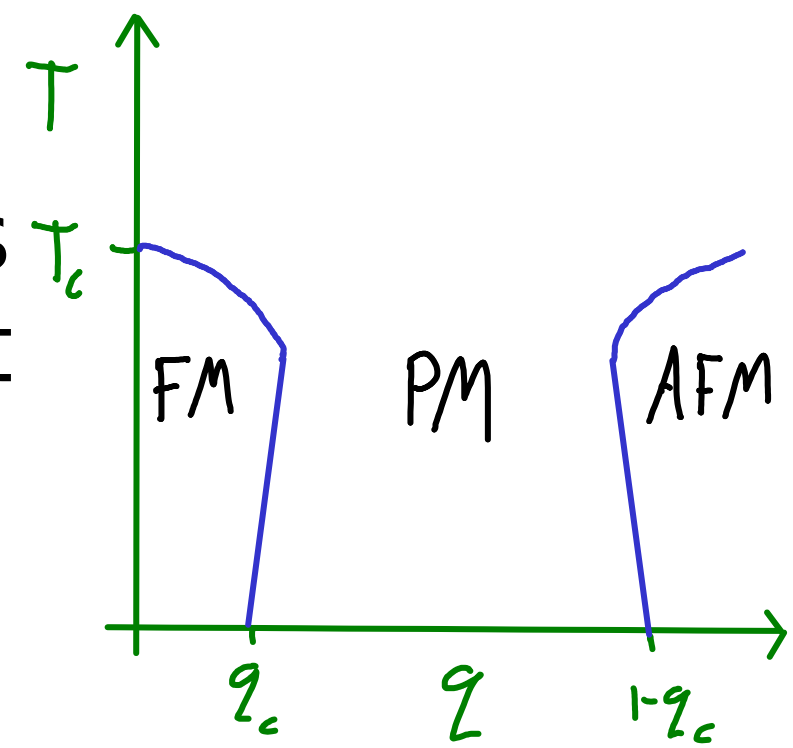
Lots of degeneracy

Seems to have lost magnetic order!



Studies (numerics and analytics) have found the following phase diagram

Note: the order parameter averages over the all different bond configs, and so over syndrome



$q_c \approx 0.11$

Ferromagnetic order in the gs disappears for $q > q_c$

Many similarities between RBIM and planar code

RBIM	PLANAR CODE
Square lattice	Square lattice
AFM bonds	bit flip errors
q	p
Frustrated plaquettes	m particles

Main difference: RBIM also has temperature T

planar code has only p

Can this be overcome?

Consider a set of AFM bonds E, and corresponding frustrated plaquettes S

For any given (Z basis) state we can determine whether each bond is satsified or not

If we consider the set of unsatisfied bonds, E', we find that they will be a set of strings with endpoints only at frustrated plaquettes (or boundaries)

The energy of a state with $N_{E'}$ unsatisfied bond is

$+J$ for each unsatisfied bond
$-J$ for each satisfied bond

$$\text{total} = JN_{E'} - J(n - N_{E'}) = 2JN_{E'} - Jn$$

But, since only energy differences matter in physics, lets rescale this such that

$$\mathcal{E}(E') = 2JN_{E'}$$

The partition function at temperature $T = 1/\beta k_B$ is then

$$Z(S) = \sum_{E' \in S} e^{-\beta \mathcal{E}(E')} = \sum_{E' \in S} e^{-\beta 2J N_{E'}}$$

E also serves as a valid set of bit flips on the planar code, with S as the corresponding anyon configuration. This has partition function

$$Z(E' \in \chi \mid S) = \sum_{E' \in \chi \cap S} \left(\frac{p}{p_0}\right)^{N_{E'}}$$

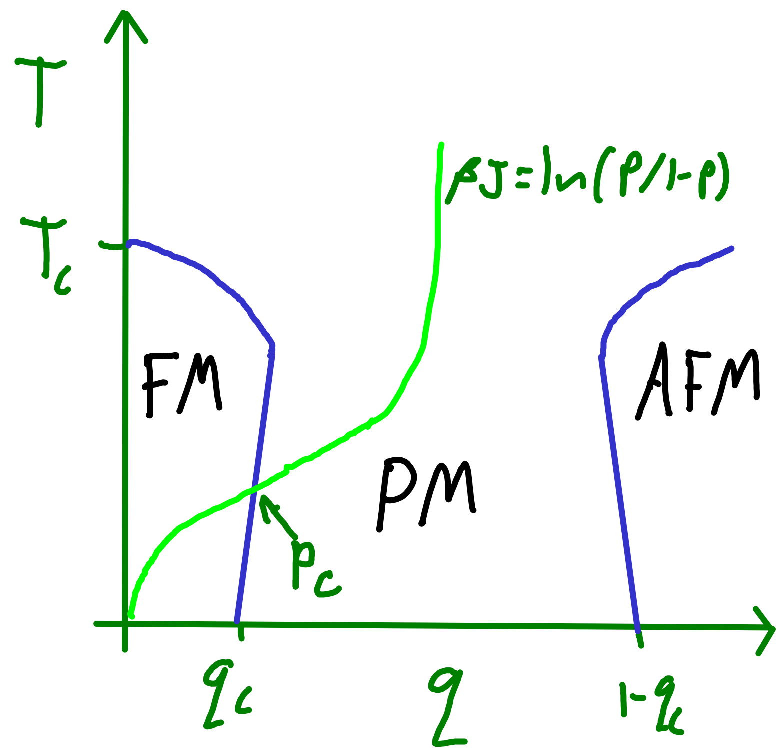
Unlike the RBIM partition function, this depends on equivalence class, but this can be ignored (see quant-ph/0110143 for more detailed mapping)

Otherwise they are identical if we set

$$q = p \qquad e^{-2J\beta} = \left(\frac{p}{1-p}\right) \therefore \beta J = -\frac{1}{2} \ln\left[p/(1-p)\right]$$

Planar code is equivalent to finite temperature RBIM, with both q and βJ given by p

Along the line of equivalence (Nishimori line) RBIM is ordered for small p and disordered for high



We can identify the ordered (FM) phase in the RBIM with the ordered

$$P_X \to 0 \text{ as } L \to \infty \quad \text{for } p < p_c$$

phase for the planar code.

The threshold error rate can be read off the phase diagram

$$p_c = q_c(\beta) = 0.1094 \pm 0.0002 \approx 11\%$$

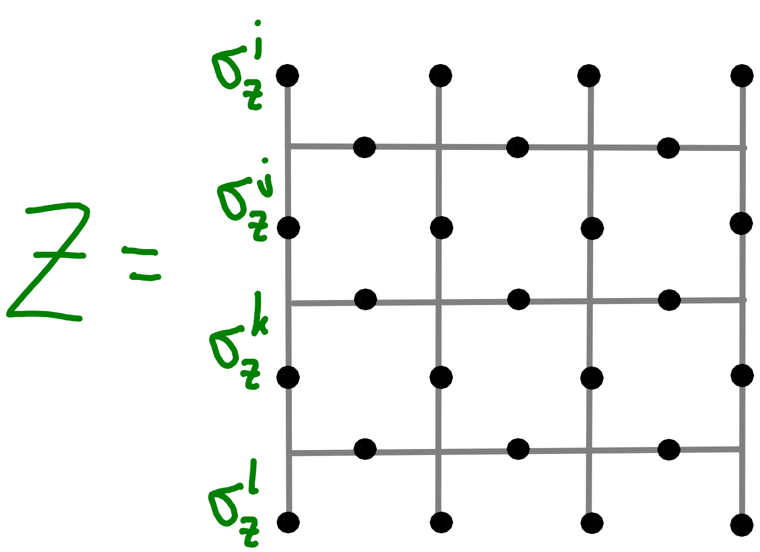
The threshold is the same for phase flip errors and the charge syndrome

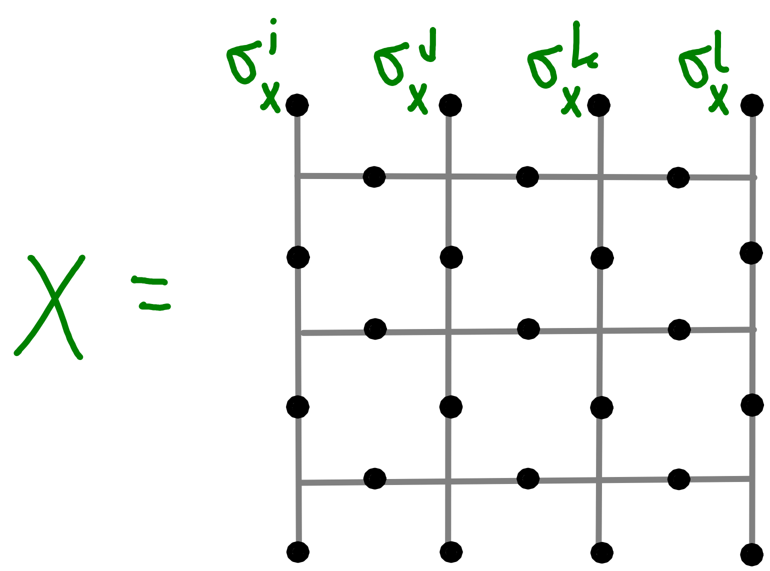
Logical Gates and Error Correction

So far we have assumed that the logical qubit is idle during error correction: it is just waiting to be used.

Can we also do logical operations during error correction? Can the error correction also correct imperfections in those operations?

Yes we can! Recall that logical X and Z are performed by single qubit rotations along a chain of physical qubits





These could all be done in a single time step, between measurement rounds

Suppose the implementation of these is noisy, adding extra noise p' on the affected qubits

Can all noise be corrected for $p + p' < p_c$?

So far we have considered only uniform noise, with the same strength on all qubits.

In this case the noise strength is $p+p'$ on L qubits, and p on the rest. Does this matter?

After making the syndrome measurements, but before decoding, we could add in extra 'fake' noise with strength p' on the qubits not affected by the logical operator

We know how these would change the syndrome, so we change it accordingly

The decoder would then deal with a syndrome created by uniform noise of strength $p+p'$, and so be highly successful for $p+p' < p_c$.

The effects of the fake noise can then be removed from the correction operator before it is applied

Probably not the best way to decode this noise, but it proves a threshold exists