Quantum Correlations

Study of correlations is very important in physics

What order/disorder does a system have?

What can it be used for?

Correlations in classical systems can be described by standard probability distributions

Quantum correlations can be more exotic. They can have

Entanglement

Quantum correlations

The other type

Separable

Classical correlations

Classical Correlations

Classical systems are described by orthogonal sets of states

They can only be described by density matrices of the form

$$\rho_{AB} = \sum_{j,k} P(j,k)\, |\psi_j \varphi_k\rangle\langle\psi_j \varphi_k| \qquad \langle\psi_j|\psi_{j'}\rangle = 0 \;\; \forall\, j \neq j'$$

$$\langle\varphi_k|\varphi_{k'}\rangle = 0 \;\; \forall\, k \neq k'$$

(SIMILARLY FOR N-PARTITE)

Correlations can be measured by the mutual information

$$I(A;B) = H[A] + H[B] - H[AB]$$

$$= S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

Also by other means (correlation functions, covariance,...)

Separable states

Consider two parties: Alice and Bob

Each have a system, no initial correlations $|0\rangle_A \otimes |0\rangle_B$

$$\left.\begin{array}{l}\text{They can communicate classically} \\ \text{They can perform arbitrary operations on} \\ \text{their qubit (Measurement, rotation, interaction} \\ \text{with local system)}\end{array}\right\} \text{LOCC}$$

They are then able to generate any state of the form

$$\rho = \sum_{i,j} p(i,j)\, |\alpha_i \beta_j\rangle\langle \alpha_i \beta_j| = \sum_{k,l} p(k,l)\, \rho_A^k \otimes \rho_B^l$$

This includes classically correlated states

Also states not possible classically, e.g. $\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|++\rangle\langle ++|$

But the correlations are nevertheless classical in origin, coming from the classical communication

For the N particle case (qubits or more generally), states generated by N party LOCC are of the form

$$\rho_{AB\ldots} = \sum_{i,j\ldots} P(i,j,\ldots)\, \rho_A^i \otimes \rho_B^j \otimes \cdots$$

These are called separable states

If they are pure they take the form

$$|\psi\rangle_{AB\ldots} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes \cdots$$

And are called product states (note they are uncorrelated)

Any separable state is a mixture of pure states

Any N partite separable state can be converted to an N partite pure state by LOCC, and vice-versa

Everything that isn't separable is beyond the reaches of our classical realm: It is entangled!

Schmidt decomposition

Consider a general state of two systems, one n dimensional, the other m dimensional, n⩽m

$$|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\,|\alpha_i \beta_j\rangle \qquad \langle\alpha_i|\alpha_j\rangle = \delta_{ij} \quad \langle\beta_i|\beta_j\rangle = \delta_{ij} \qquad \mathcal{H}_A = \mathbb{C}^n \qquad \mathcal{H}_B = \mathbb{C}^m$$

Can we write this in a simpler way? How about

$$|\psi\rangle = \sum_{i=1}^{n} |\alpha_i\rangle \otimes \sum_{j=1}^{m} c_{ij}\,|\beta_j\rangle = \sum_{i=1}^{n} \tilde{c}_i\,|\alpha_i \tilde{\alpha}_i\rangle$$

$$|\tilde{\alpha}_i\rangle = \frac{1}{\tilde{c}_i}\sum_{j=1}^{m} c_{ij}\,|\beta_j\rangle\,, \qquad \tilde{c}_i = \sqrt{\sum_{j=1}^{m} c_{ij}\,c_{ij}^*}$$

This has only one sum and all coefficients are real (and positive). However, in general

$$\langle\tilde{\alpha}_i|\tilde{\alpha}_j\rangle \neq \delta_{jk}$$

Can we also find a way to make these orthogonal?

The properties of this decomposition depend on the basis we use to express the state of the first qubit.

Which is the best basis to use? Which basis would be any better than any other? How about the eigenbasis of ρ_A ?

Find the density matrix for the smaller system and diagonalize it. Use this as the $|\alpha_j\rangle$ basis

$$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum_{i=1}^{n} p_i |\alpha_i\rangle\langle\alpha_i| , \qquad |\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} |\alpha_i \beta_j\rangle$$

The coefficients can then be related to the eigenvalues

$$p_i = \sum_j |\langle\psi|\alpha_i \beta_j\rangle|^2 = \sum_j c_{ij} c_{ij}^*$$

Using the same trick as before we can reduce the two sums to one

$$\tilde{c}_i = \sqrt{\sum_{j=1}^{m} c_{ij} c_{ij}^*} = \sqrt{p_i} , \qquad |\tilde{\alpha}_i\rangle = \frac{1}{\tilde{c}_i} \sum_{j=1}^{m} c_{ij} |\beta_j\rangle = \frac{1}{\sqrt{p_i}} \sum_{j=1}^{m} c_{ij} |\beta_j\rangle$$

$$\therefore \quad |\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i} |\alpha_i \tilde{\alpha}_i\rangle$$

These states will be orthogonal. To prove this, we can simply do the partial trace in this basis

$$\rho = \sum_{i,j} \sqrt{p_i p_j} \; |\alpha_i\rangle\langle\alpha_j| \otimes |\tilde{\alpha}_i\rangle\langle\tilde{\alpha}_j|$$

$$\rho_A = \sum_{i,j} \sqrt{p_i p_j} \; |\alpha_i\rangle\langle\alpha_j| \otimes tr\left(|\tilde{\alpha}_i\rangle\langle\tilde{\alpha}_j|\right)$$

$$= \sum_{i,j} \sqrt{p_i p_j} \; |\alpha_i\rangle\langle\alpha_j| \; \langle\tilde{\alpha}_j|\tilde{\alpha}_i\rangle$$

Since we know, by definition, that $\rho_A = \sum_{i=1}^{n} p_i |\alpha_i\rangle\langle\alpha_i|$

Clearly $\langle\tilde{\alpha}_j|\tilde{\alpha}_i\rangle = \delta_{ij}$

So they are indeed orthogonal!

This is the Schmidt decomposition:

Any bipartite pure state can be expressed

smallest dimension

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}\, |a_i \tilde{a}_i\rangle$$

Real ORTHONORMAL

It gives us some nice results for the reduced density matrices

$$\rho_A = \sum_i p_i |a_i\rangle\langle a_i| \qquad \rho_B = \sum_i p_i |\tilde{a}_i\rangle\langle \tilde{a}_i|$$

SAME EIGENVALUES

Note that this means

$$S(\rho_A) = S(\rho_B) = -\sum_i p_i \ln p_i$$

Overal purity requires equal entropies

The state is entangled for $S(\rho_A) > 0$

Note that this decomposition only applies to bipartite pure states

Doesn't apply to mixed

Doesn't apply to multipartite (though you can always do a bipartition)

Example: $|\psi\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$

TRIPARTITE SCHMIDT ? $\rho_A = \rho_B = \rho_C = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$|\psi'\rangle = \sqrt{\frac{2}{3}}\,|000\rangle + \sqrt{\frac{1}{3}}\,|111\rangle \neq |\psi\rangle$ NOT POSSIBLE

BIPARTITE SCHMIDT : $|\psi\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle \otimes |01\rangle + |0\rangle \otimes |10\rangle + |1\rangle \otimes |00\rangle\right)$

$= \sqrt{\frac{2}{3}}\,|0\rangle \otimes \underbrace{\left(|01\rangle + |10\rangle\right)}_{\sqrt{2}} + \sqrt{\frac{1}{3}}\,|1\rangle \otimes |00\rangle$ Success!

Assessing Separability

Given a N-partite pure state, it is easy to determine whether or not it is a product state

Given an N-partite mixed state, it is not so easy to determine whether it is separable

Can $\rho_{ABC\ldots}$ be expressed $\rho_{AB\ldots} = \sum_{i,j\ldots} q(i,j,\ldots)\, \rho_A^i \otimes \rho_B^j \otimes \cdots$?

Cannot be determined in poly N time

Inefficient methods are impractical for large N

However, clues can be found more quickly

Separability criterion

Entanglement measures

Entanglement witnesses

Peres-Horodecki separability criterion

The transpose of a matrix, expressed in Dirac notation, is

$$M = \sum_{ij} m_{ij} |i\rangle\langle j| , \quad M^T = \sum_{ij} m_{ij} |j\rangle\langle i| = \sum_{ij} m_{ji} |i\rangle\langle j| ,$$

The results of the transpose depend on the basis you use (trivial for eigenbasis, for example)

The transpose of a pure state always gives a valid pure state

$$|\psi\rangle = \sum_j c_j |\alpha_j\rangle , \quad |\psi\rangle\langle\psi| = \sum_{j,k} c_j c_k^* |\alpha_j\rangle\langle\alpha_k|$$

$$\left(|\psi\rangle\langle\psi|\right)^T = \sum_{j,k} c_k c_j^* |\alpha_j\rangle\langle\alpha_k| = |\psi^*\rangle\langle\psi^*| , \quad |\psi^*\rangle = \sum_j c_j^* |\alpha_j\rangle$$

So the transpose of a mixed state is also a valid mixed state

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| , \quad \rho^T = \sum_j p_j |\psi_j^*\rangle\langle\psi_j^*|$$

For a bipartite density matrix $\rho_{AB} = \sum_{ijkl} C_{ijkl} |i\rangle\langle k| \otimes |j\rangle\langle l|$

We define the partial transposes

$$\rho_{AB}^{T_A} = \sum_{ijkl} C_{ijkl} |k\rangle\langle i| \otimes |j\rangle\langle l| \qquad \rho_{AB}^{T_B} = \sum_{ijkl} C_{ijkl} |i\rangle\langle k| \otimes |l\rangle\langle j|$$

The results will clearly still be Hermitian and trace 1, as required for a density matrix, but are they still positive?

If the state is separable

$$\rho_{AB} = \sum_{i,j} p_{ij}\, \rho_A^i \otimes \rho_B^j \quad \therefore \quad \rho^{T_A} = \sum_{i,j} p_{ij}\, \rho_A^{i\,T} \otimes \rho_B^j$$

Transpose of a DM is positive (also hermitian and trace 1)

Partial traces of separable states are therefore positive

What about entangled states?

There exist entangled states whose partial transpose is not positive

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \qquad \rho_{AB} = |\psi\rangle\langle\psi| = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$\therefore \rho_{AB}^{T_B} = \frac{1}{2}\left(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\right)$$

CONSIDER $|01\rangle - |10\rangle$ $\qquad \rho_{AB}^{T_B}\left(|01\rangle - |10\rangle\right) = |10\rangle - |01\rangle = -\left(|01\rangle - |10\rangle\right)$

$\therefore$ $|01\rangle - |10\rangle$ IS EIGENSTATE WITH EIGENVALUE -1

$\therefore$ $\rho_{AB}^{T_B}$ IS NOT +VE

So any state that does not have a positive partial transpose (PPT) must be entangled

Do PPT entangled states exist?

For 2x2, 2x3 or 3x2 density matrices, no PPT entangled states exist: this criterion finds all entangled states

Otherwise PPT entangled states are possible. But they are a bit odd

(Bipartite) Entanglement Measures

How entangled are two systems A and B?

First consider pure states

$|\psi\rangle = |00\rangle$ NOT ENTANGLED

$|\psi\rangle = \sqrt{1-\varepsilon}\,|00\rangle + \sqrt{\varepsilon}\,|11\rangle$ A BIT ENTANGLED

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ VERY ENTANGLED

Entanglement is a correlation

No other types of correlation is present in pure states

So let's us MI $I(A;B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$

Entropy of pure states is zero $S(\rho_{AB}) = 0$

From Schmidt $S(\rho_A) = S(\rho_B)$

$\therefore\ I(A;B) = 2\,S(\rho_A)$

For bipartite pure states, the entropy alone is a good measure. We call this the entropy of entanglement

$$\mathcal{E}(|\psi\rangle) = S(\rho_A)$$

Given this measure, what are the most entangled states of two qubits?

Bell basis: $|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ $\quad |\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$

$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$ $\quad |\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

In each case $\rho_A = \rho_B = \frac{1}{2}\mathbb{1} \quad \therefore \quad S(\rho_A) = 1$

This is the maximum value for qubits

Other maximally entangled two qubit states are local unitary rotations of these

$$U_A \otimes U_B |\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|\alpha_0 \beta_0\rangle + |\alpha_1 \beta_1\rangle\right), \quad \langle\alpha_i|\alpha_j\rangle = \langle\beta_i|\beta_j\rangle = \delta_{ij}$$

Bell basis is an orthonormal basis for 2 qubits, but an entangled basis rather than a product one

They are equivalent up to local Paulis

$$\sigma_x^1 |\phi^+\rangle = \sigma_x^2 |\phi^+\rangle = |\psi^+\rangle, \quad \sigma_z^1 |\phi^+\rangle = \sigma_z^2 |\phi^+\rangle = |\phi^-\rangle, \quad \sigma_z^1 \sigma_x^1 |\phi^+\rangle = \ldots = |\psi^-\rangle$$

For mixed states, good entanglement measure would satisfy the following conditions

$E(\rho_{AB}) = 0$ if ρ_{AB} is separable

$E(\rho_{AB}) = 1$ if ρ_{AB} is a Bell basis state (for qubits)

$E(U_A \otimes U_B \, \rho_{AB} \, U_A^\dagger \otimes U_B^\dagger) = E(\rho_{AB})$

$E(\rho'_{AB}) \leq E(\rho_{AB})$ if ρ'_{AB} is ρ_{AB} after LOCC

Most straightforward solution: Entanglement of formation

$$E_F(\rho_{AB}) = \min_{\{|\psi_i\rangle_{AB}\}} \sum_i p_i \, \mathcal{E}(|\psi_i\rangle_{AB}) \quad \text{for} \quad \rho_{AB} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|_{AB}$$

For a given decomposition into pure states, average entropy of entanglement is found. This is then minimized over all decompositions

Hard to calculate in general!

Concurrence

For states of two qubits, there is a closed form. This gives rise to another measure: the concurrence

Given ρ, find: $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$

Now diagonalize : $\sqrt{\sqrt{\rho}\,\tilde{\rho}\sqrt{\rho}}$

Label the eigenvalues: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$

Then : $C(\rho) = \max[\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0]$

This is related to the entanglement of formation via

$$E_F = H\left(\frac{1+\sqrt{1-C^2}}{2}\right) \quad \therefore \quad E_F = 0 \iff C = 0$$

$$E_F = 1 \iff C = 1$$

$$0 < E_F < 1 \iff 0 < C < 1$$

Non Locality and Contextuality

Bell's theorem is the classic test of quantum non-locality and contextuality

Instead, let's take a look at Hardy's paradox

A pair of objects are sent to Alice and Bob.

They consider two possible measurements, M and m

Both measurements have binary outcomes 0 and 1

They both do M measurements for many samples and find that they never both get the outcome 1

If one measures M and the other m, they find that
m=1 always implies M=1

So what happens if both measure m?

Assume that the objects don't communicate, and so neither knows how the other was measured (non-contextual)

If Alice gets m=1, we know that Bob would get M=1 if he measured M

If Bob gets m=1, we know that Alice would get M=1 if she measured M

Therefore, if Alice and Bob both get m=1, they both would have gotten M=1 if they measured M

Since this never happens, both getting m=1 won't happen either

If Alice and Bob find that both getting m=1 is possible, it is proof that the measurements are contextual: Alice's results depend on what Bob is doing

Consider the objects to be qubits in an entangled state

$$|\psi\rangle = a|00\rangle + b|01\rangle + b|10\rangle \qquad a,b \in \mathbb{R}$$

The measurement M is measurement in the Z basis

The measurement m is in the basis $|m_0\rangle, |m_1\rangle$

$$|0\rangle = \mu_0|m_0\rangle + \mu_1|m_1\rangle \qquad |1\rangle = \mu_1|m_0\rangle - \mu_0|m_1\rangle \qquad \mu_0,\mu_1 \in \mathbb{R}$$

With μ_0 and μ_1 such that $\langle\psi|0m_1\rangle = 0$

$$|\psi\rangle = a\mu_0|0m_0\rangle + b\mu_1|0m_0\rangle + b\mu_0|1m_0\rangle \qquad \therefore\ a\mu_1 = b\mu_0$$

$$+\ a\mu_1\underline{|0m_1\rangle} - b\mu_0\underline{|0m_1\rangle} + b\mu_1|1m_1\rangle$$

And μ_0 and μ_1 such that $\langle\psi|m_1 0\rangle = 0$

$$|\psi\rangle = a\mu_0|m_0 0\rangle + b\mu_0|m_0 1\rangle + b\mu_1|m_0 0\rangle \qquad \therefore\ a\mu_1 = b\mu_0$$

$$+\ a\mu_1\underline{|m_1 0\rangle} + b\mu_1|m_1 1\rangle - b\mu_0\underline{|m_1 0\rangle}$$

For simplicity, take the case of $a = b = \frac{1}{\sqrt{3}}$ $\therefore$ $\mu_0 = \mu_1 = \frac{1}{\sqrt{2}}$

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left(2\,|0 m_0\rangle + |1 m_0\rangle + |1 m_1\rangle\right)$$

$$= \frac{1}{\sqrt{6}}\left(2\,|m_0 0\rangle + |m_0 1\rangle + |1 m_0\rangle\right)$$

When both measure m (which is the X basis)

$$|\psi\rangle = \frac{1}{\sqrt{12}}\Big(2\,|m_0 m_0\rangle + |m_0 m_0\rangle + |m_0 m_1\rangle$$

$$2\,|m_1 m_0\rangle - |m_1 m_0\rangle + |m_1 m_1\rangle\Big)$$

$$= \frac{1}{\sqrt{12}}\left(3\,|m_0 m_0\rangle + |m_0 m_1\rangle + |m_1 m_0\rangle + |m_1 m_1\rangle\right)$$

The result m=1 for both is possible, and occurs with prob 1/12

This shows that QM is contextual

The qubits don't communicate (unless they do)

This effect is due to the inherently non-local nature of quantum correlations

I don't understand it either! But I know how to make use of it.