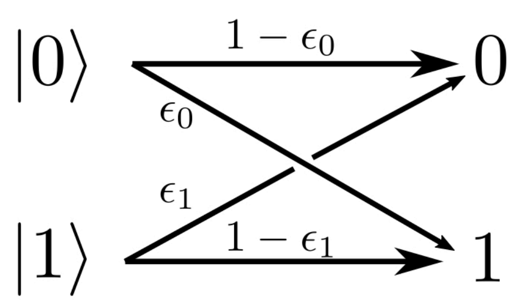
Exercise 3

Due November 25 before noon

1 Readout errors and stochastic matrices

At the end of a quantum computation, the state of the qubits is measured. The act of measurement is a classical communication problem — the computer is sending messages to the user. Let's say that at the end of the computation, the probability that the qubit is in state $|1\rangle$ is p, while the probability that the qubit is in $|0\rangle$ is $1-p$. A noisy measurement device is then used to make an inference — "0" or "1" — about the state of the qubit, with errors ϵ_0 and ϵ_1 described by an asymmetric bitflip channel:



(a) Write down the matrix S that relates the vector of pre-measurement probabilities, $P = (1-p, p)^\top$, to the vector of post-measurement probabilities, $P' = (1-p', p')^\top$. The matrix should satisfy $P' = SP$. (**2 marks**)

(b) The measurement fidelity F gives the probability of correctly identifying the qubit state in a given measurement. Consider the case of equal priors, $p = 1/2$. What is F in terms of $\mathrm{Tr}\, S$? Hint: The fidelity can be computed as $F = P(\text{“0”}, 0) + P(\text{“1”}, 1)$, where $P(\text{“}j\text{”}, j)$ is the *joint* probability of the qubit being in state $|j\rangle$ and of correctly inferring the outcome "j". (**2 marks**)

(c) A matrix M is stochastic if (1) $M_{ij} \geq 0\ \forall i, j$ and (2) $\sum_i M_{ij} = 1\ \forall j$. If S^{-1} is *not* stochastic, then the channel is irreversible and information has been lost. Compute S^{-1} for $\epsilon_0 = \epsilon_1 = \epsilon$. Find all values of ϵ for which S^{-1} is a stochastic matrix. Explain why your result makes sense. (**4 marks**)

2 Control power in quantum teleportation

Let's consider a three-party quantum teleportation scheme involving Alice, Bob, and Charlie. Alice has a single-qubit state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ that she wants to send to Bob. Alice, Bob, and Charlie share a three-qubit GHZ state given by $|\Psi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. Charlie is an agent of chaos who may or may not cooperate. Let's see whether Alice and Bob can still benefit from shared entanglement without Charlie's help.

(a) Alice performs a Bell measurement on $|\psi\rangle$ and her portion of the shared GHZ state. She obtains one of four measurements, given by

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{1}$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{2}$$

Calculate the post-measurement state of Bob and Charlie's qubits conditioned on the outcome of Alice's Bell measurement. (Make sure this state is properly normalized.) Record your answers in the following table (**2 marks**):

Alice's outcome	State of Bob and Charlie's qubits
$\|\Phi^+\rangle$	
$\|\Phi^-\rangle$	
$\|\Psi^+\rangle$	
$\|\Psi^-\rangle$	

(b) Charlie is in a good mood and decides to help out. He now performs an X-basis measurement of his qubit and communicates the outcome to Bob. Building on your answer from (a), write down the state of Bob's qubit following Alice's Bell measurement and Charlie's X-basis measurement. Identify the correction operator Bob needs to apply to recover the original state $|\psi\rangle$. Assume a basis ordering in which $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, and record your answers in the table below (**4 marks**).

Alice's outcome	Charlie's outcome	State of Bob's qubit	Correction
$\|\Phi^+\rangle$	$\|+\rangle$		
$\|\Phi^+\rangle$	$\|-\rangle$		
$\|\Phi^-\rangle$	$\|+\rangle$		
$\|\Phi^-\rangle$	$\|-\rangle$		
$\|\Psi^+\rangle$	$\|+\rangle$		
$\|\Psi^+\rangle$	$\|-\rangle$		
$\|\Psi^-\rangle$	$\|+\rangle$		
$\|\Psi^-\rangle$	$\|-\rangle$		

Note that *with* Charlie's cooperation, Bob can recover Alice's state with perfect fidelity.

(c) Charlie is in a bad mood and wants no part in Alice and Bob's shenanigans. He measures his qubit but does not record the outcome. Without Charlie's help, Bob does his best to reconstruct Alice's original state $|\psi\rangle$: Having received the outcome of Alice's Bell measurement, he applies the following correction operator to his qubit conditioned on Alice's outcome:

$$\left|\Phi^+\right\rangle \to \mathbb{1}, \tag{3}$$

$$\left|\Phi^-\right\rangle \to Z, \tag{4}$$

$$\left|\Psi^+\right\rangle \to X, \tag{5}$$

$$\left|\Psi^-\right\rangle \to ZX. \tag{6}$$

Building on your results from (a), apply the correction given above, then trace over Charlie's qubit to model Charlie's lack of cooperation. In other words, evaluate the state ρ_{B} of Bob's qubit as

$$\rho_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{C}}(\rho_{\mathrm{BC}}), \tag{7}$$

where ρ_{BC} is the state of Bob and Charlie's qubits with the appropriate correction operator already applied. Note that tracing over Charlie's qubit amounts to discarding the outcome of Charlie's measurement. (**3 marks**)

(d) For an initial state $|\psi\rangle$, $F_\psi = \langle\psi|\rho_{\mathrm{B}}|\psi\rangle$ gives the fidelity of the teleportation achieved without Charlie's help. Average F_ψ over the single-qubit Haar measure $d\psi$ to obtain the state-averaged fidelity $\bar{F}$ of the teleportation protocol:

$$\bar{F} = \int d\psi \, \langle\psi|\rho_{\mathrm{B}}|\psi\rangle. \tag{8}$$

(**3 marks**) Hint: This average can be performed as an average over the polar and azimuthal coordinates θ and ϕ as

$$\int d\psi \, (\cdots) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, (\cdots). \tag{9}$$

The factor of 4π in the denominator accounts for the 4π solid angle of the Bloch sphere. You may want to express a and b in terms of θ and ϕ.

The highest attainable fidelity for a "classical" teleportation protocol is $\bar{F} = 2/3$. Such a strategy involves Alice making a measurement of her qubit (which is *not* entangled with Bob's qubit), sending the outcome to Bob over a classical communication channel, and having Bob perform operations on another qubit in an attempt to reconstruct Alice's original state. If Charlie withholds his measurement outcome, does quantum teleportation with the GHZ state provide an advantage over classical resources? (**1 mark**)

(e) Let's say we wanted to implement such a teleportation scheme experimentally, and to measure the state-averaged fidelity $\bar{F}$. Having the initial state $|\psi\rangle$ parameterized by a continuum of angles (θ, ϕ) would make evaluating $\bar{F}$ somewhat inconvenient in such a scenario. (What θ, ϕ do we choose? How finely do we need to sample the Bloch sphere, and would a finite mesh even reproduce the statistics of the Haar measure $d\psi$?) Express $\langle\psi|\rho_{\mathrm{B}}|\psi\rangle$ as a polynomial $p(\boldsymbol{x})$ in two variables a and b, i.e. $\boldsymbol{x} = (a, b)^\top$, and compute (**2 marks**)

$$\frac{1}{|X|} \sum_{\boldsymbol{x} \in X} p(\boldsymbol{x}). \tag{10}$$

Here, $|X| = 6$ is the cardinality of the set X, i.e. the number of elements in X, while X itself is given by

$$X = \left\{ (1,0), (0,1), \frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,i), \frac{1}{\sqrt{2}}(1,-i) \right\}. \tag{11}$$

Does this quantity look familiar? If you were an experimentalist, what initial states $|\psi\rangle$ would you prepare in order measure the average fidelity of your teleportation protocol *without* averaging over all $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$? (**1 mark**)