Quantum Information 2019
Sheet 7

Shor Code

The Shor (or 9 Qubit) code is a method for quantum error correction based on the classical repetition code. First a logical qubit is stored in three physical qubits using the encoding

$$|+\rangle_3 = |+++\rangle\,, \;\; |-\rangle_3 = |---\rangle\,.$$

This protects against errors that try to flip $|+\rangle_3$ to $|-\rangle_3$, and vice-versa. But the corresponding errors for the Z basis states

$$|0\rangle_3 = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle),\;\; |1\rangle_3 = \frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle)$$

become more likely. To deal with this we take three of these logical qubits and use them (like the original physical qubits) to encode a single logical qubit. This uses the encoding:

$$|0\rangle_9 = |0\rangle_3 \otimes |0\rangle_3 \otimes |0\rangle_3\,, \;\; |1\rangle_9 = |1\rangle_3 \otimes |1\rangle_3 \otimes |1\rangle_3\,.$$

The end result is then a code that stores one logical qubit in 9 physical qubits, with stabilizer states

$$|0\rangle_9 = \left[\frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle)\right]^{\otimes 3}, \;\; |1\rangle_9 = \left[\frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle)\right]^{\otimes 3}$$

a) Find operators that act as X and Z on the logical qubit. What are the minimal number of qubits these act on?

b) Suppose σ_x errors occur independently on each qubit with probability p_x. What is the probability P_x that a logical X occurs after syndrome measurement and error correction? For simplicity you can determine this only up to lowest order in p_x.

c) Similarly, what is the probability P_z of Z errors, given that σ_z errors occur with probability p_z? For simplicity you can determine this only up to lowest order in p_z.

Concatenated Shor Code

In order to increase the performance of a code we can use the concept of concatenation. We will now consider this process for the Shor code.

Let us describe physical qubits as level-0 qubits, and suppose we have n of them. We can use these to encode $n/9$ logical qubits, which we call level-1 qubits. These will have lower probabilities for noise than the level-0 qubits, but maybe not as low as we require. We can then use the level-1 qubits as if they were physical qubits, using them to encode $n/9^2$ level-2 qubits. This procedure can then be continued as many times as required, with the level-$(l-1)$ qubits always used as the physical qubits of the Shor codes that encode the level-l qubits.

a) In order to encode a single level-l qubit, for arbitrary l, what is the number $n(l)$ of level-0 qubits required?

b) The standard Shor code has distance $d = 3$. What is the distance of a level-l concatenated Shor code?

c) Show that $p_x^{(l)}$ decays exponentially with $n(l)^\beta$ when $p_x < 1/27$, and find β.

This is a proof of the 'threshold theorem' for this code and error model. As long as the physical noise rate p_x is below the threshold value of $1/27$ (and p_z is below its threshold of $1/9$), concatenation of the Shor code can achieve arbitrarily low error rates.