**Quantum Information** 2018
**Sheet 9**

## 1. Toric Code

The planar code is defined on a plane with rough and smooth boundary conditions. But we could instead wrap the $L \times L$ square lattice around a torus and have periodic boundary conditions. Then the code would be translationally invariant, and all $A_s$ and $B_p$ stabilizers would be four qubit operators.

a) The parameter $L$ counts the number of plaquettes along each direction. Show that $n = 2L^2$, where $n$ is the number of qubits.

b) Show that the number of plaquette operators is $L^2$, but that the number of *independent* plaquette operators is $L^2 - 1$. Show the same thing for the vertex operators.

c) How many logical qubits, $k$, can be stored in the stabilizer space?

d) Define logical $X$ and $Z$ operators for these logical qubits. Note that these are not uniquely defined. However, as with any stabilizer code, you will know that your logical operators are a valid choice if they satisfy the following conditions.

1. Logical Pauli operators must commute with all stabilizers.
2. Logical Pauli operators for the same logical qubit anticommute: $\{X_j, Z_j\} = 0$.
3. Logical Pauli operators for different logical qubits commute: $[X_j, Z_k] = 0$.

## 2. Wen plaquette model

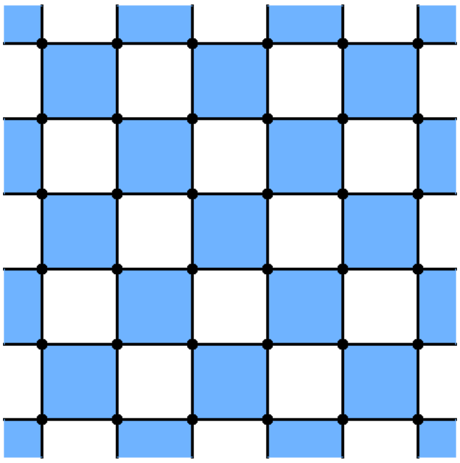
The Wen plaquette model is defined on a square lattice with periodic boundary conditions. However, unlike the toric code, the physical qubits reside on the vertices.

The size of the code is measured by the linear size $L_W$ which counts the number of qubits (or, equivalently, plaquettes) along each side. When $L_W$ is even it is possible to bicolour the plaquettes. When $L_W$ is odd this is not possible. An example of the lattice with $L_W = 6$ is shown below.

Stabilizers are defined on plaquettes only and take the form $W_p = \sigma_x^1 \sigma_z^2 \sigma_x^3 \sigma_z^4$, where the spins of a plaquette are numbered clockwise from the top left.

a) Show that all stabilizers commute.

b) The number of logical qubits, $k$, depends on whether $L_W$ is odd or even. Find $k$ in both cases.



**3. Equivalence of the two**

Consider two stabilizer codes, one with stabilizers $\{S_j\}$ and the other with stabilizers $\{S'_j\}$. Consider also a unitary $U$ composed of single qubit unitaries $U_j$ on each qubit $j$.

$$U = \bigotimes_j U_j$$

Such an operator is known as a local unitary.

Two codes are said to be local unitary equivalent if there exists a $U$ such that such that

$$\{S'_j\} = \{U S_j U^\dagger\}.$$

a) Show that the Toric Code is local unitary equivalent to the Wen plaquette model with even $L_W$.

b) Show that they are not equivalent if $L_W$ is odd.

*Hint:* Use the single qubit unitary $H$ for which

$$H\sigma_x H^\dagger = \sigma_z, \;\; H\sigma_z H^\dagger = \sigma_x.$$