Bits

Information theory treats all information as strings of symbols

As humans we typically use letters, numbers and punctuation

Why?

- We can quite easily remember ~100 symbols
- It means that the kind of numbers and words we usually deal with aren't too long

In IT, these symbols must be stored in different states of the physical system used to build a device

The more symbols used, the more complicated the device. Better to use only a few, even if it makes the numbers long

Because of this, IT usually uses binary. Two symbols: 0 and 1

z.B. 101, 11001, 1011001, 1001011100, 11111111011

Qubits

The quantum version of a bit is a 2D quantum system

Bits 2 POSSIBLE STATES : 0, 1

Qubits 2 D HILBERT SPACE : $\mathcal{H} = \mathbb{C}^2$

BASIS STATES : $|0\rangle, |1\rangle$

Bit string 01100

Multipartite qubit state $|0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle = |01100\rangle \in \mathcal{H}^{\otimes n}$

Mathematically (and sometimes physically) they are identical to spin-1/2 particles

The difference:

Qubits are a more general concept

The methods used are inspired by information theory

Basis states for single qubits

We've already met one orthonormal basis for a single qubit

$$|0\rangle,\ |1\rangle \quad : \quad \langle 0|0\rangle = \langle 1|1\rangle = 1$$

$$\langle 0|1\rangle = \langle 1|0\rangle = 0$$

$|0\rangle, |1\rangle$ BASIS,

COMPUTATIONAL BASIS,

Z BASIS

There are infinitely many more, but two others are commonly used

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\langle +|+\rangle = \langle -|-\rangle = 1 \qquad \langle +|-\rangle = \langle -|+\rangle = 0$$

$$\langle 0|+\rangle = \langle 0|-\rangle = \frac{1}{\sqrt{2}} \qquad \langle 1|+\rangle = -\langle 1|-\rangle = \frac{1}{\sqrt{2}}$$

$|+\rangle, |-\rangle$ BASIS

X BASIS

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad |\circlearrowleft\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$\langle \circlearrowright|\circlearrowright\rangle = \langle \circlearrowleft|\circlearrowleft\rangle = 1 \ldots \quad etc$$

Y BASIS

These can be represented as vectors

Usually we use the basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

So

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad |\circlearrowright\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |\varphi\rangle = \alpha|1\rangle + \beta|+\rangle + \gamma|\circlearrowright\rangle = \begin{pmatrix} \alpha + (\beta+\gamma)/\sqrt{2} \\ (\beta - i\gamma)/\sqrt{2} \end{pmatrix}$$

And for matrices

$$\langle 0|\psi\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = a \qquad |-\rangle\langle\psi| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} a & b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a & b \\ -a & -b \end{pmatrix}$$

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} = m_{00}|0\rangle\langle 0| + m_{01}|0\rangle\langle 1| + m_{10}|1\rangle\langle 0| + m_{11}|1\rangle\langle 1|$$

Pauli Matrices

A matrix with eigenvalues λ_j and eigenstates $|\lambda_j\rangle$ can be written in spectral form

$$M = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|$$

Clearly a hermitian matrix requires real eigenvalues

$$M = M^\dagger \Rightarrow \lambda_j = \lambda_j^* \quad \forall j$$

And a unitary matrix must have magnitude 1 eigenvalues

$$UU^\dagger = \mathbb{1} \Rightarrow \lambda_j \lambda_j^* = 1 \quad \forall j$$

Matrices that are both hermitian and unitary can therefore only have eigenvalues $\lambda_j = \pm 1$

Qubit matrices of this form are nontrival if they have one of each possible eigenvalues

For $\langle\alpha|\beta\rangle = 0$ $\qquad |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| = \mathbb{1}$, $\quad -|\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta| = -\mathbb{1}$

but $|\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|$ is something interesting

Such matrices are very useful in QI

The most important ones are defined using the three orthonormal bases

$$\sigma_x = |+\rangle\langle+| - |-\rangle\langle-| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = |↻\rangle\langle↻| - |↺\rangle\langle↺| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For notational convenience, we can also include

$$\sigma_0 = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These are the Pauli matrices

They are related to each other by

$$\sigma_x \sigma_y = i\,\sigma_z$$

$$\sigma_\alpha \sigma_{\alpha+1} = i\,\sigma_{\alpha+2} \qquad \alpha \in \{x, y, z\} \qquad x+1 = y,\ y+1 = z,\ z+1 = x$$

They also have the property $\sigma_\alpha^{\,2} = \sigma_0$

Projectors

Projectors are operators for which each eigenvalue is 0 or 1

We say that it 'projects onto' eigenspace with eigenvalue 1

The projector acts as identity on that subspace and annihilates all else

z.B. $\mathbb{P} = |00\rangle\langle 00| + |11\rangle\langle 11|$

$$\mathbb{P}\left(a|00\rangle + b|11\rangle\right) = a|00\rangle + b|11\rangle$$

$$\mathbb{P}\left(c|01\rangle + d|10\rangle\right) = 0$$

$$\mathbb{P}\left(a|00\rangle + c|01\rangle + d|10\rangle + b|11\rangle\right) = a|00\rangle + b|11\rangle$$

The projectors can act on single states

z.B. $\mathbb{P} = |+\rangle\langle +|$

or higher dimensional subspaces (as above)

In any case, they square to themselves $\mathbb{P}^2 = \mathbb{P} \Rightarrow \mathbb{P}^n = \mathbb{P}$

Bloch Sphere

A qubit state is a vector in $\mathbb{C}^2$

$$|\psi\rangle = \alpha e^{i\gamma}|0\rangle + \beta e^{i\delta}|1\rangle$$

4 REAL NUMBERS

But it is also a ray: global phase is unphysical

$$|\psi\rangle = \alpha|0\rangle + e^{i\phi}\beta|1\rangle$$

$$\phi = \delta - \gamma$$

3 REAL NUMBERS

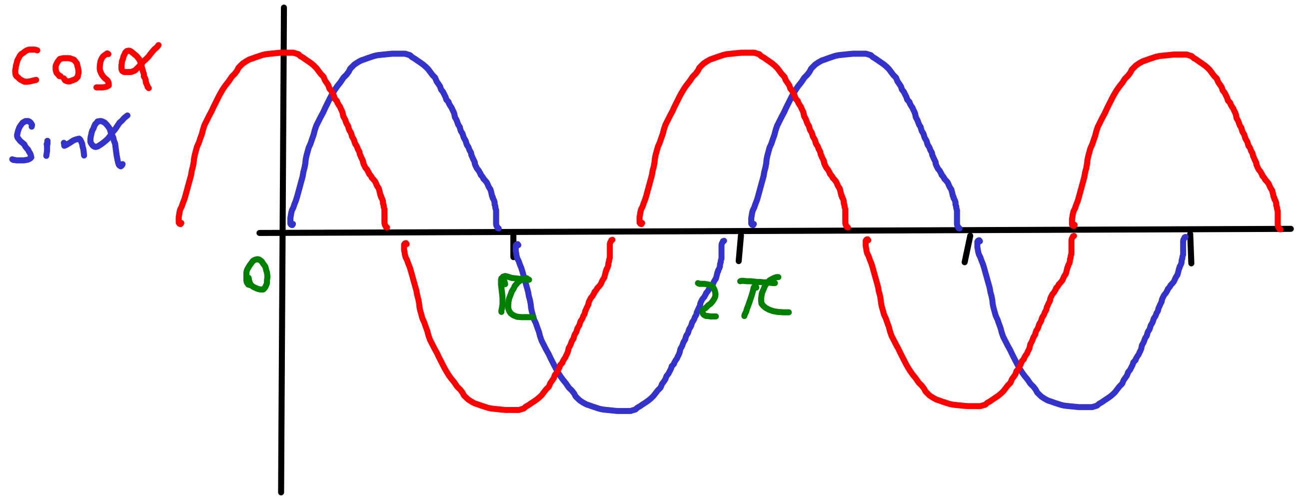
It is also normalized: $\alpha^2 + \beta^2 = 1$

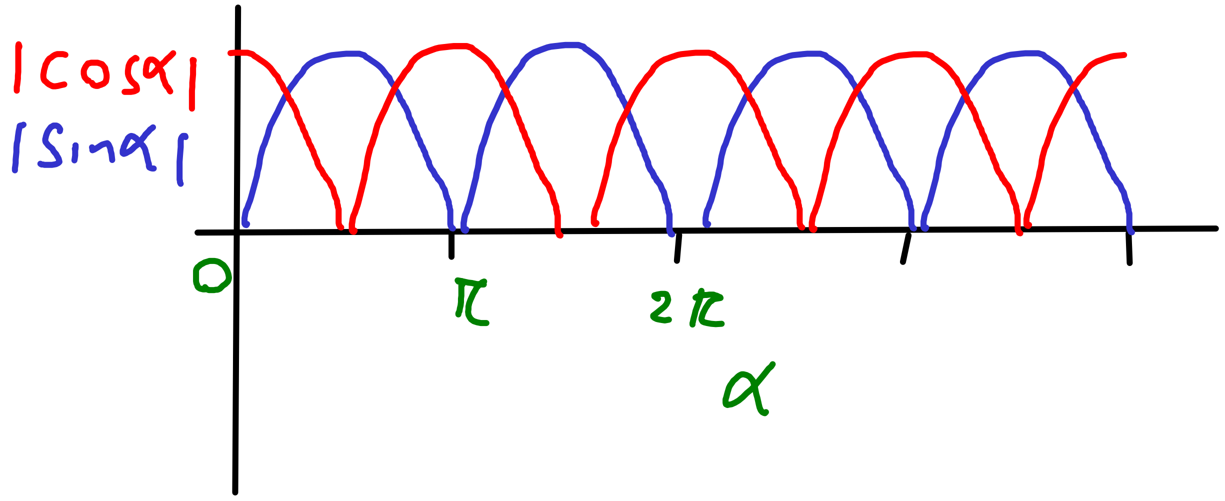
$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$\theta = 2\cos^{-1}(\alpha) = 2\sin^{-1}(\beta)$$

2 REAL NUMBERS

$0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi$





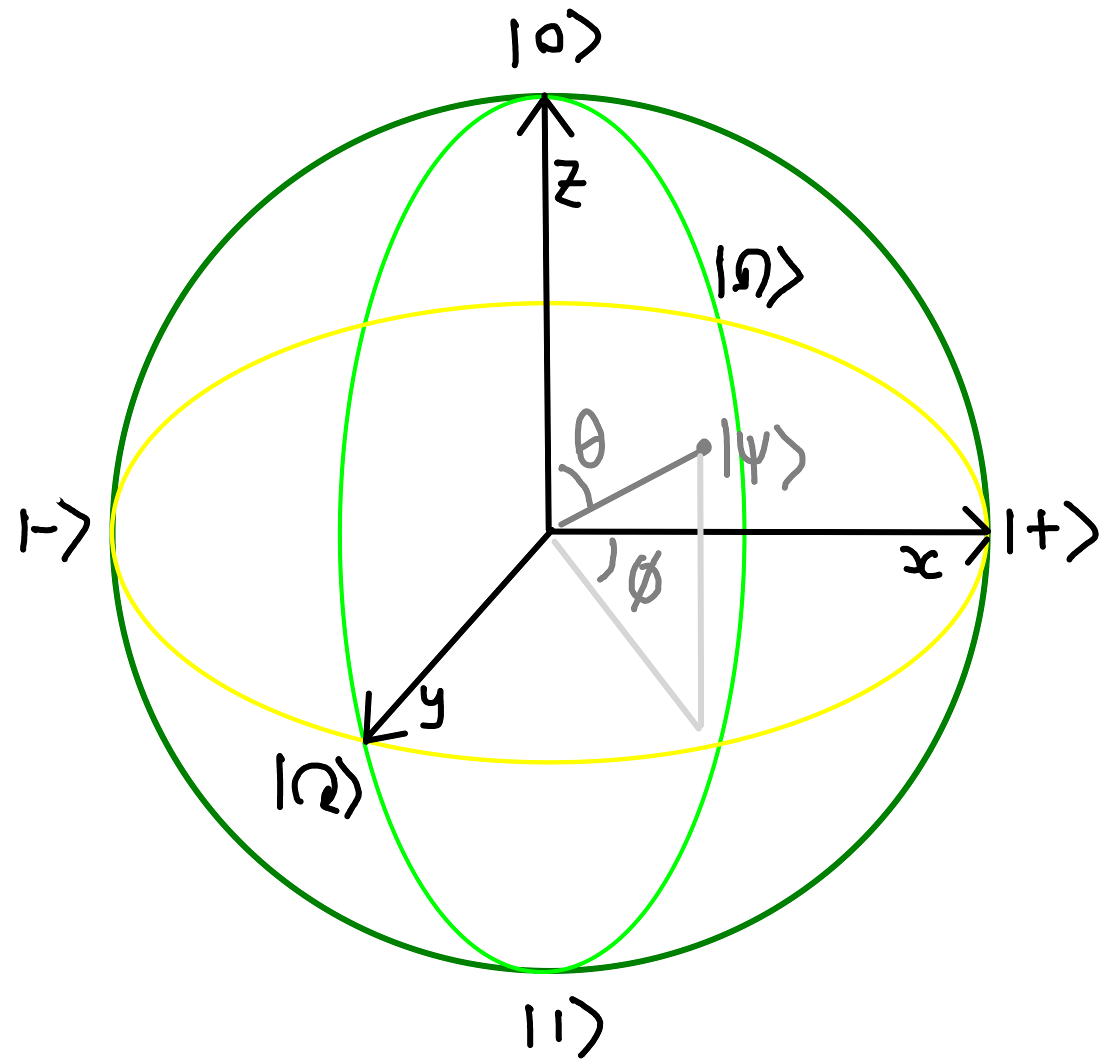
Unit vector in $\mathbb{R}^3$ is also specified by 2 real numbers

$$\begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} \qquad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi$$

We can think of qubit states as points on a sphere

$$\left.\begin{aligned} |\psi\rangle &= \cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\tfrac{\theta}{2}\right)|1\rangle \\ \vec{\psi} &= \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} \end{aligned}\right\} \qquad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi$$

The Bloch sphere



Density Matrices (Part 1)

Though we usually represent states as kets

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

We can also do so with the projector

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a^* & b^* \end{pmatrix} = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} = \begin{pmatrix} 1-|b|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$$

These can be decomposed into Pauli operators

$$\sigma_0 = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_x = |+\rangle\langle +| - |-\rangle\langle -| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_y = |\circlearrowright\rangle\langle\circlearrowright| - |\circlearrowleft\rangle\langle\circlearrowleft| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\rho_\psi = \frac{1}{2}\left[\sigma_0 + (|a|^2 - |b|^2)\sigma_z + 2\,\mathrm{Re}(a^*b)\,\sigma_x + 2\,\mathrm{Im}(a^*b)\,\sigma_y\right]$$

$$= \frac{1}{2}\sum_\alpha \langle\sigma_\alpha\rangle\sigma_\alpha$$

$$\rho_\psi = \frac{1}{2} \sum_\alpha \langle \sigma_\alpha \rangle \sigma_\alpha$$

The coefficients are the expectation value of the Pauli operator observables

$$\langle O \rangle \equiv \langle \psi | O | \psi \rangle = \langle \psi | O | \psi \rangle\langle \psi | \psi \rangle$$

$$= \sum_k \langle \psi_k | O \rho_\psi | \psi_k \rangle, \quad \langle \psi_k | \psi \rangle = \delta_k$$

$$\mathrm{tr}\, M \equiv \sum_k m_{kk} = \sum_k \langle \psi_k | M | \psi_k \rangle$$

$$\therefore \quad \langle O \rangle = \mathrm{tr}(O \rho_\psi)$$

Note that $\quad \mathrm{tr}(\sigma_\alpha \sigma_\beta) = 2\delta_{\alpha,\beta}$

$$\therefore \quad \langle \sigma_\alpha \rangle = \mathrm{tr}(\sigma_\alpha \rho)$$

They are also the Cartesian co-ordinates for the Bloch sphere

A corresponding Pauli decomposition is possible for n qubit states

$$|\psi\rangle \in (\mathbb{C}^2)^{\otimes n} \qquad |\psi\rangle\langle\psi| = \frac{1}{2^n} \sum_{\alpha_1, \alpha_2, \dots, \alpha_n} \langle \sigma_{\alpha_1} \sigma_{\alpha_2} \cdots \sigma_{\alpha_n} \rangle \, \sigma_{\alpha_1} \sigma_{\alpha_2} \cdots \sigma_{\alpha_n}$$

Mixed States

We seen that, for a state $|\psi\rangle$ and observable O, the expectation value can be written

$$\langle O \rangle = tr(O \rho_\psi), \qquad \rho_\psi = |\psi\rangle\langle\psi|$$

This can be used when we know the state of the system. But what if we don't? What if we are given a qubit and told it is in state $|\psi_j\rangle$ with probability p_j. How do we mathematically describe such a state?

Clearly $$\langle O \rangle = \sum_j p_j \, tr(O \rho_{\psi_j})$$

Since the trace is linear, this can be expressed

$$\langle O \rangle = tr\left(O \sum_j p_j \rho_{\psi_j}\right)$$

So we can simply replace the set of projectors ρ_{ψ_j} with

$$\rho = \sum_j p_j \rho_{\psi_j}$$

This is also a density matrix

As such, it can also be expressed in terms of Pauli operators

$$\rho_{\psi_j} = \frac{1}{2} \sum_{\alpha} \langle \sigma_\alpha \rangle_j \sigma_\alpha \qquad \therefore \qquad \rho = \frac{1}{2} \sum_{\alpha} \langle \sigma_\alpha \rangle \sigma_\alpha$$

$$\langle \sigma_\alpha \rangle = \sum_j p_j \langle \sigma_\alpha \rangle_j$$

$$\rho = \frac{1}{2^n} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n} \langle \sigma_{\alpha_1} \sigma_{\alpha_2} \cdots \sigma_{\alpha_n} \rangle \sigma_{\alpha_1} \sigma_{\alpha_2} \cdots \sigma_{\alpha_n}$$

This more general form of state is called a 'mixed state'

The projectors are called 'pure states'

Density Matrices (Part 2)

Note that we are not free to put just any numbers in for the Pauli operator expectation values

$$\rho = \frac{1}{2} \sum_{\alpha} \langle \sigma_\alpha \rangle \sigma_\alpha$$

Some correspond to unphysical states, e.g.

$$\langle \sigma_z \rangle = 1000 \quad \text{or} \quad \langle \sigma_x \rangle = \langle \sigma_x \rangle = \langle \sigma_z \rangle = 1$$

The conditions for a valid density matrix are

It is Hermitian (probabilities are real)

$tr(|\psi\rangle\langle\psi|) = 1$ (probabilities sum to 1)

$tr(|\psi\rangle\langle\psi| \, \mathbb{P}) \geq 0 \quad \forall \mathbb{P}$ (probabilities are non-negative)

The first two conditions are straightforward, but what about the last?

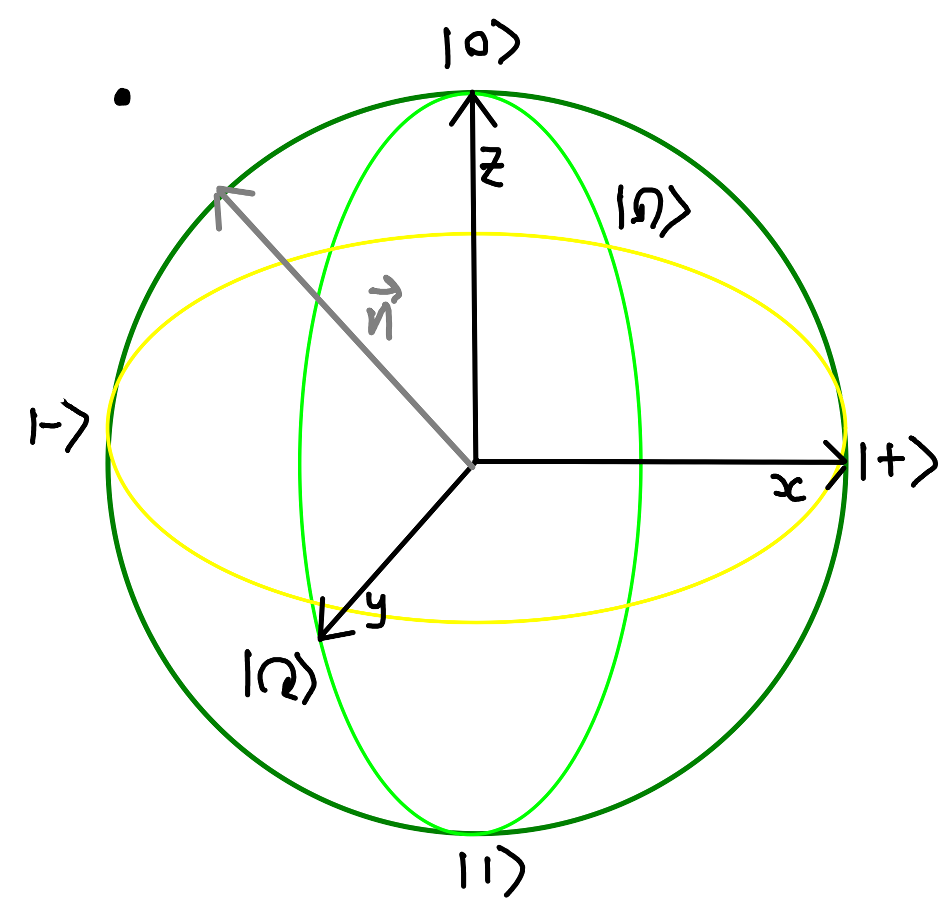
Consider the σ_z expectation value

$$\langle \sigma_z \rangle = p(0) - p(1) \quad \therefore \quad p(1) = \langle \sigma_z \rangle + p_0$$

$$\therefore \quad \langle \sigma_z \rangle > 1 \implies p(1) < 0$$

If this or any other expectation value is >1 or <-1 it implies that a probability will be negative

Note that these would correspond to points outside of the Bloch sphere



For any other point outside of the Bloch sphere we can define a rotated Pauli matrix along the direction that points at it

$$\sigma = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z, \quad \vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}, \quad n_x^2 + n_y^2 + n_z^2 = 1$$

For which $\langle \sigma \rangle > 1$

So for a single qubit, the last condition can be interpreted as

- states cannot be outside of the Bloch sphere

- $\langle \sigma \rangle^2 \leq 1 \quad \forall\ \sigma = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z,$

$$\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}, \quad n_x^2 + n_y^2 + n_z^2 = 1$$

The latter can be expressed more succinctly

$$\langle \sigma \rangle^2 = \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2$$

$$\therefore \quad \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \leq 1$$

Pure states are those for which the equality holds

Mixed states are the interior of the Bloch sphere