|

a) $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle \qquad \beta = \sqrt{1-\alpha^2}$

$$\therefore \rho = \alpha^2|00\rangle\langle 00| + \alpha\beta|00\rangle\langle 11| + \alpha\beta|11\rangle\langle 00| + \beta^2|11\rangle\langle 11|$$

$$\therefore \rho^{T_B} = \alpha^2|00\rangle\langle 00| + \alpha\beta|01\rangle\langle 10| + \alpha\beta|10\rangle\langle 01| + \beta^2|11\rangle\langle 11|$$

$$= \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & \alpha\beta & 0 \\ 0 & \alpha\beta & 0 & 0 \\ 0 & 0 & 0 & \beta^2 \end{pmatrix}$$

Find the eigenvalues. Note that it is block diagonal

$$\rho^{T_B} = \alpha^2 \mathbb{1} \oplus \beta^2 \mathbb{1} \oplus \alpha\beta\sigma^x$$

So eigenvalues are

$$\alpha^2, \ \beta^2, \ \alpha\beta, \ -\alpha\beta$$

negative $\forall \ \alpha, \beta \neq 0$

b) State is $|\phi\rangle\langle\phi|$ with prob. p and $\sigma_x^1|\phi\rangle\langle\phi|\sigma_x^1$ with prob. $1-p$, so

$$\rho = p\,\sigma_x^1|\phi\rangle\langle\phi|\sigma_x^1 + (1-p)|\phi\rangle\langle\phi|$$

$$= p\left(\alpha^2|10\rangle\langle 10| + \alpha\beta|10\rangle\langle 01| + \alpha\beta|01\rangle\langle 10| + \beta^2|01\rangle\langle 01|\right)$$
$$+(1-p)\left(\alpha^2|00\rangle\langle 00| + \alpha\beta|00\rangle\langle 11| + \alpha\beta|11\rangle\langle 00| + \beta^2|11\rangle\langle 11|\right)$$

$$\rho^{T_B} = p\left(\alpha^2|10\rangle\langle 10| + \alpha\beta|00\rangle\langle 11| + \alpha\beta|11\rangle\langle 00| + \beta^2|01\rangle\langle 01|\right)$$
$$+(1-p)\left(\alpha^2|00\rangle\langle 00| + \alpha\beta|10\rangle\langle 01| + \alpha\beta|01\rangle\langle 10| + \beta^2|11\rangle\langle 11|\right)$$

$$= \begin{pmatrix} (1-p)\alpha^2 & & & p\alpha\beta \\ & p\beta^2 & (1-p)\alpha\beta & \\ & (1-p)\alpha\beta & p\alpha^2 & \\ p\alpha\beta & & & (1-p)\beta^2 \end{pmatrix}$$

2 block diagonal 2×2 matrices

$$\rho^{T_B} = \begin{pmatrix} (1-p)\alpha^2 & p\alpha\beta \\ p\alpha\beta & (1-p)\beta^2 \end{pmatrix} \oplus \begin{pmatrix} p\beta^2 & (1-p)\alpha\beta \\ (1-p)\alpha\beta & p\alpha^2 \end{pmatrix}$$

can be diagonalized normally, or we can recall that if trace 1, hermitian matrices are expressed in the form

$$\frac{1}{2}\left(\mathbb{1} + \langle\sigma_x\rangle\sigma_x + \langle\sigma_y\rangle\sigma_y + \langle\sigma_z\rangle\sigma_z\right)$$

They have +ve eigenvalues iff

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 \leq 1$$

So lets cast the problem in this form

$$\begin{pmatrix} (1-p)\alpha^2 & p\alpha\beta \\ p\alpha\beta & (1-p)\beta^2 \end{pmatrix} = (1-p)\begin{pmatrix} \alpha^2 & \bar{p}\alpha\beta \\ \bar{p}\alpha\beta & \beta^2 \end{pmatrix}, \quad \bar{p} = \frac{p}{1-p}$$

$$= (1-p)\frac{1}{2}\left(\mathbb{1} + (2\alpha^2-1)\sigma_z + \bar{p}\alpha\beta\sigma_x\right)$$

$$\begin{pmatrix} p\beta^2 & (1-p)\alpha\beta \\ (1-p)\alpha\beta & (p\alpha^2) \end{pmatrix} = p\begin{pmatrix} \beta^2 & \bar{p}^{-1}\alpha\beta \\ \bar{p}^{-1}\alpha\beta & \alpha^2 \end{pmatrix}$$

$$= p\,\frac{1}{2}\left(\mathbb{1} + (2\beta^2-1)\sigma_z + \frac{\alpha\beta}{\bar{p}}\sigma_x\right)$$

The conditions for +ve eigenvalues are then

1. $(2\alpha^2-1)^2 + (\bar{p}\alpha\beta)^2 \leq 1$

2. $(2\beta^2-1) + \left(\frac{\alpha\beta}{\bar{p}}\right)^2 \leq 1$

rearranging in terms of $\bar{p}$

① $\bar{p}^2 \leq (\alpha\beta)^{-2} - (2\alpha^2-1)^2$

② $\bar{p}^2(2\beta^2-1) + (\alpha\beta)^2 \leq \bar{p}^2$

$2\bar{p}^2(\beta^2-1) + (\alpha\beta)^2 \leq 0$

$\sqrt{2}\,\bar{p}\,\alpha \leq -\alpha\beta$

$\therefore\ \bar{p} \leq -\frac{\beta}{\sqrt{2}}$

Looks like ① is the important one, but this probably needs checking

2

a) $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$

$\therefore \rho_A = \begin{pmatrix} \alpha^2 & \\ & \beta^2 \end{pmatrix}$

$\therefore E(\rho_A) = S(\rho_A) = -\alpha^2 \log \alpha^2 - \beta^2 \log \beta^2$

b) C is found by diagonalizing

$R = \sqrt{\sqrt{\rho}\,\tilde{\rho}\,\sqrt{\rho}}$, $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$

Note that $\rho = \sqrt{\rho} = \rho^*$ in the case considered

$\therefore R = \sqrt{\rho \sigma_y \otimes \sigma_y \rho \sigma_y \otimes \sigma_y \rho}$

For the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$

$$C(\rho) = \max[\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0]$$

c) Hopefully it comes out obviously

3

Alice has a classical variable X and Bob a classical variable Y

$$\left.\begin{array}{l} X: \; x_j \\ Y: \; y_k \end{array}\right\} \; p(x_j, y_k) = P(j,k)$$

If alice has x_j, she prepares her system in state ρ_A^j. Similar for Bob.

Final state of quantum systems is then

$$\rho_{AB} = \sum_{jk} P(j,k)\, \rho_A^j \otimes \rho_B^k$$

as required.