$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

a) X basis states: $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

$U|+\rangle = |0\rangle, \quad U|-\rangle = |1\rangle$

Y basis states $|\circlearrowright\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$

$U|\circlearrowright\rangle = |\circlearrowleft\rangle, \quad U|\circlearrowleft\rangle = |\circlearrowright\rangle$

Z basis states $|0\rangle, \; |1\rangle$

$U|0\rangle = |+\rangle \quad U|1\rangle = |-\rangle$

b) Tell them that any single qubit unitary is a rotation around some axis by some angle. They just need to find these.

Since $U^2 = \mathbb{1}$, the angle is π (or 2π, but this clearly isn't true)

By noticing that $U = \frac{1}{\sqrt{2}}\left(\sigma_x + \sigma_z\right)$, the axis is that with unit vector $\vec{u} = \frac{1}{\sqrt{2}}\vec{x} + \frac{1}{\sqrt{2}}\vec{z}$

c) $U\sigma_x U^\dagger = \sigma_z \qquad U\sigma_z U^\dagger = \sigma_x$

$U\sigma_y U^\dagger = -\sigma_y$

$$\therefore \quad \frac{1}{2}\left(\mathbb{1} + \langle\sigma_x\rangle\sigma_x + \langle\sigma_y\rangle\sigma_y + \langle\sigma_z\rangle\sigma_z\right)$$

$$\rightarrow \frac{1}{2}\left(\mathbb{1} + \langle\sigma_z\rangle\sigma_x - \langle\sigma_y\rangle\sigma_y + \langle\sigma_x\rangle\sigma_z\right)$$

The x and z expectation values exchange, and the y one inverts

d) They should be able to infer

$F\sigma_x F^\dagger = \sigma_y, \quad F\sigma_y F^\dagger = \sigma_x, \quad F\sigma_z F^\dagger = -\sigma_z$

$G\sigma_z G^\dagger = \sigma_y, \quad G\sigma_y G^\dagger = \sigma_z, \quad G\sigma_x G^\dagger = -\sigma_x$

$H\sigma_x H^\dagger = \sigma_z, \quad H\sigma_z H^\dagger = \sigma_x, \quad H\sigma_y H^\dagger = -\sigma_y$

So • $H = U$ from (a) – (c)

- F is similarly a π rotation around $\frac{1}{\sqrt{2}}\vec{x} + \frac{1}{\sqrt{2}}\vec{y}$
- G " " " " " $\frac{1}{\sqrt{2}}\vec{y} + \frac{1}{\sqrt{2}}\vec{z}$