Quantum Information 2018
Sheet 8

1. Kraus operators

a) Consider single qubit noise that applies a σ_x with probability p_x, a σ_y with probability p_y and a σ_z with probability p_z. The probability that nothing is applied is $1 - p_x - p_y - p_z$. What are the Kraus operators for this?

b) The single qubit noise operator

$$\varepsilon(\rho) = (1-p)\rho + p\,\frac{1}{2}\sigma_0$$

can be expressed in the same form as part (a) with suitably chosen p_x, p_y and p_z. Find these.

c) Consider the following Hamiltonian that interacts two qubits,

$$H = \frac{1}{2}\left(\sigma_0 \otimes \sigma_0 + \sigma_x \otimes \sigma_0 + \sigma_0 \otimes \sigma_z - \sigma_x \otimes \sigma_z\right). \tag{1}$$

If initial state of the second qubit is $|+\rangle$, what are the Kraus operators for the resulting process on the first qubit after a time t.

d) As (c), but with initial state $|0\rangle$ for the second qubit.

2. Concatenated Shor Code

In order to increase the performance of a code we can use the concept of concatenation. We will now consider this process for the Shor code.

Let us describe physical qubits as level-0 qubits, and suppose we have n of them. We can use these to encode $n/9$ logical qubits, which we call level-1 qubits. These will have lower probabilities for noise than the level-0 qubits, but maybe not as low as we require. We can then use the level-1 qubits as if they were physical qubits, using them to encode $n/9^2$ level-2 qubits. This procedure can then be continued as many times as required, with the level-$(l-1)$ qubits always used as the physical qubits of the Shor codes that encode the level-l qubits.

a) In order to encode a single level-l qubit, for arbitrary l, what is the number $n(l)$ of level-0 qubits required?

b) The standard Shor code has distance $d = 3$. What is the distance of a level-l concatenated Shor code?

Using the result from Sheet 4 (the last time we saw the Shor code), we can see that the bit flip error probabilities $p_x^{(l-1)}$ and $p_x^{(l)}$ are related by

$$p_x^{(l)} < 27 \left(p_x^{(l-1)} \right)^2 , \tag{2}$$

where $p_x^{(0)} = p_x$.

c) Show that $p_x^{(l)}$ decays exponentially with $n(l)^\beta$ when $p_x < 1/27$, and find β.

This is a proof of the 'threshold theorem' for this code and error model. As long as the physical noise rate p_x is below the threshold value of $1/27$ (and p_z is below its threshold of $1/9$), concatenation of the Shor code can achieve arbitrarily low error rates.