Quantum Information 2018
Sheet 3

1. Schumacher compression

Consider a qubit density matrix ρ with eigenvalues q_0 and q_1 and eigenstates $\left|\tilde{0}\right\rangle$ and $\left|\tilde{1}\right\rangle$ respectively.

Consider a k-dimensional subspace spanned by eigenstates of $\rho^{\otimes N}$, with projector $\mathbb{P}_k$. We wish to determine what value of k is required such that

$$\mathrm{tr}(\mathbb{P}_k \rho^{\otimes N}) = 1 - \epsilon, \quad \lim_{N\to\infty} \epsilon = 0. \tag{1}$$

We can then ignore all parts of the state that do not lie in the subspace.

The k-dimensional subspace can be thought of as the space of K_N^Q qubits, where $K_N^Q = \log_2 k$. So we are compressing the original N qubits down to K_N^Q.

In lecture it was claimed that condition (1) can be satisfied with

$$K_N^Q = N(H(q_1) + \delta)$$

for arbitrarily small but finite δ. Prove this.

2. Pauli operatiors as a basis for matrices

(a) Show that any 2×2 matrix can be expressed in the form

$$M = a\sigma^0 + b\sigma^x + c\sigma^y + d\sigma^z$$

(b) Under what conditions will M be unitary?

3. Set of Pauli operators is complete

The non-trivial Pauli operators satisfy the following mutual anticommutation property,

$$\{\sigma^j, \sigma^k\} = 0, \;\; j \neq k = \{x, y, z\}.$$

Show that these is no fourth Pauli, i.e. no operator σ^w such that

$$\{\sigma^w, \sigma^k\} = 0, \;\; k = \{x, y, z\}.$$

4. Positivity of the density matrix

(a) Show that

$$\mathrm{tr}(\rho_\psi \mathbb{P}) \geq 0$$

for any projector $\mathbb{P}$ and density matrix for a pure state ρ_ψ (which is also a projector).

(b) Show the same for a general density matrix, ρ.