2

General Matrix: $M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$

$$= m_{00}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + m_{01}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + m_{10}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + m_{11}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Can express each of these in terms of Paulis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_0 + \sigma_z$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_0 - \sigma_z$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\sigma_0 + \sigma_z)\sigma_x = \sigma_x + i\sigma_y$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\sigma_0 - \sigma_z)\sigma_x = \sigma_x - i\sigma_y$$

3 Suppose such a σ_w does exist

$$\sigma_w \sigma_z = -\sigma_z \sigma_w \quad \Rightarrow \quad \sigma_w \sigma_z \sigma_w^\dagger = -\sigma_z$$

$$\therefore \; \sigma_w |0\rangle\langle 0| \sigma_w^\dagger - \sigma_w |1\rangle\langle 1| \sigma_w^\dagger = -|0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\therefore \; \sigma_w |0\rangle = e^{i\theta} |1\rangle$$
$$\sigma_w |1\rangle = e^{i\varphi} |0\rangle$$
for some $\theta, \varphi \in \mathbb{R}$

This gives us enough info to write down a general matrix for σ_w

$$\sigma_w = \sigma_w \mathbb{1} = \sigma_w (|0\rangle\langle 0| + |1\rangle\langle 1|) = (\sigma_w |0\rangle)\langle 0| + (\sigma_w |1\rangle)\langle 1|$$
$$= e^{i\theta} |0\rangle\langle 1| + e^{i\varphi} |1\rangle\langle 0| = \begin{pmatrix} 0 & e^{i\theta} \\ e^{i\varphi} & 0 \end{pmatrix}$$

Now impose the condition $\sigma_w = \sigma_w^\dagger$

$$\begin{pmatrix} 0 & e^{i\theta} \\ e^{i\varphi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{-i\theta} & 0 \end{pmatrix} \quad \Rightarrow \quad \sigma_w = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$, we find

$$\sigma_w = \cos\theta\, \sigma_x + i \sin\theta\, \sigma_y$$

Now impose the condition $\{\sigma_x, \sigma_w\} = 0$

$$\{\sigma_x, \sigma_w\} = 0 \Rightarrow \sigma_x \sigma_w \sigma_x = -\sigma_w$$

$$\sigma_x \sigma_w \sigma_x = \cos\theta \, \sigma_x - i \sin\theta \, \sigma_y \qquad \therefore \quad \cos\theta = 0$$
$$-\sigma_w = -\cos\theta \, \sigma_x - i \sin\theta \, \sigma_y$$

This leads to the solutions $\sigma_w = \pm \sigma_y$

Obviously, neither will satisfy the final condition

$$\{\sigma_x, \sigma_w\} = 0$$

So no fourth Pauli exists

4

a) $\rho_\psi = |\psi\rangle\langle\psi| \quad \therefore \quad tr(\rho_\psi \mathbb{P}) = tr(|\psi\rangle\langle\psi|\mathbb{P}) = \langle\psi|\mathbb{P}|\psi\rangle$

$\mathbb{P} = \sum_j |\varphi_j\rangle\langle\varphi_j|, \quad \langle\varphi_j|\varphi_k\rangle = \delta_{jk} \quad \therefore tr(\rho_\psi \mathbb{P}) = \sum_j \langle\psi|\varphi_j\rangle\langle\varphi_j|\psi\rangle$

$\langle\psi|\varphi_j\rangle\langle\varphi_j|\psi\rangle = \langle\psi|\varphi_j\rangle\left(\langle\psi|\varphi_j\rangle\right)^* \geq 0$ since this is a general property of complex numbers

$\therefore tr(\rho_\psi \mathbb{P}) \geq 0$, as required

b) ρ can be written in terms of its eigenvalues and eigenvectors

$$\rho = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|, \qquad \langle\lambda_j|\lambda_k\rangle = \delta_{jk}$$

Consider the projector $\mathbb{P} = |\lambda_j\rangle\langle\lambda_j|$, for which $tr(\rho\mathbb{P}) = \lambda_j$

The property $tr(\rho\mathbb{P}) \geq 0$ then implies $\lambda_j \geq 0 \;\forall j$, as required