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a) With probability p_x we get state $\sigma_x \rho \sigma_x$, etc. So final state is described by the density matrix

$$p_0 \rho + p_x \sigma_x \rho \sigma_x + p_y \sigma_y \rho \sigma_y + p_z \sigma_y \rho \sigma_y$$

$$p_0 = 1 - p_x - p_y - p_z$$

So when expressed in terms of Kraus operators

$$\mathcal{E}(\rho) = \sum_m E_m \rho E_m^\dagger$$

We get four of them $E_0 = \sqrt{p_0}\, \sigma_0$

$$E_x = \sqrt{p_x}\, \sigma_x, \quad E_y = \sqrt{p_y}\, \sigma_y, \quad E_z = \sqrt{p_z}\, \sigma_z$$

b) Check out N+C 8.3.4, and tell the students to do so too.

First let's consider a different question: What p_x, p_y and p_z would result in

$$\mathcal{E}(\rho) = \frac{1}{2} \mathbb{1}$$

For the density matrix $\frac{1}{2}\mathbb{1}$ we find

$$\langle \sigma_z \rangle = 0$$

Given a density matrix ρ with $\langle \sigma_z \rangle \neq 0$ we can create ρ' with $\langle \sigma_z \rangle = 0$ as follows

$$\rho' = \frac{1}{2}\rho + \frac{1}{2}\sigma_x \rho \sigma_x$$

For this $\rho'_{00} = \frac{1}{2}\rho_{00} + \frac{1}{2}\rho_{11}$

$$\rho'_{11} = \frac{1}{2}\rho_{11} + \frac{1}{2}\rho_{00} = \rho'_{00}$$

$$\therefore \langle \sigma_z \rangle = \rho'_{00} - \rho'_{11} = 0$$

Similarly applying the other Pauli ops with prob. $\frac{1}{2}$ will 'unbias' $\langle \sigma_y \rangle$ and $\langle \sigma_x \rangle$

So

$$\frac{1}{2}\mathbb{1} = \frac{1}{4}\rho + \frac{1}{4}\sigma_x \rho \sigma_x + \frac{1}{4}\sigma_y \rho \sigma_y + \frac{1}{4}\sigma_z \rho \sigma_z$$

Then for $\mathcal{E}(\rho) = (1-p)\rho + p\mathbb{1}$

We find $p_x = p_y = p_z = p/4$

$$p_0 = 1-p + \frac{p}{4} = 1 - \frac{3p}{4}$$

c) $H = \frac{1}{2}(11 + X1 + 1Z - XZ)$

where $X1 = \sigma_x \otimes \mathbb{1}$, etc.

$$\boxed{\begin{array}{l}\text{Hint: } H \text{ has eigenvalues } \pm 1 \text{, so} \\ e^{-iHt} = \sigma_0 \otimes \sigma_0 \cos(t) - i H \sin(t)\end{array}}$$

Kraus ops are then defined (as in lectures)

$$E_+ = \langle + | e^{-iHt} | + \rangle$$

$$= \sigma_0 \cos t - i \frac{1+X}{2} \sin t$$

$$= \sigma_0 \cos t - i \,|+\rangle\langle+|\, \sin t$$

$$E_- = \langle - | e^{-iHt} | + \rangle$$

$$= -i \,|-\rangle\langle-|\, \sin t$$

Note that this causes the state to be diagonal in the X basis at $t = \frac{\pi}{2}$

d)
$$E_0 = \langle 0 | e^{-iHt} | 0 \rangle = \sigma_0$$
$$E_1 = \langle 1 | e^{-iHt} | 0 \rangle = 0$$

So no errors happen ever!