1

a) $\rho_A = (a^2+ab)|0\rangle\langle 0| + (b^2+ab)|1\rangle\langle 1|$

$\rho_B = (a^2+ab)|+\rangle\langle +| + (b^2+ab)|-\rangle\langle -|$

note that $tr(\rho) = 1 \Rightarrow a+b=1$

$\therefore \rho_A = a|0\rangle\langle 0| + b|1\rangle\langle 1|, \ \rho_B = a|+\rangle\langle +| + b|-\rangle\langle -|$

b) A must have $|0\rangle$ with prob a and $|1\rangle$ with prob b. Same for B, but for $|+\rangle$ and $|-\rangle$.

A and B are independent in ρ, but let's correlate them to make a pure state

$|\psi\rangle = a|0+\rangle + b|1-\rangle$

2 (Note: this is a lazy proof)

The prob. of n $|\tilde{1}\rangle$'s and $N-n$ $|\tilde{0}\rangle$'s is

$$P(n) = \binom{N}{n} q_1^n q_0^{N-n}$$

as $N \to \infty$, this becomes a normal distribution. For $N \gg 1$ it is a good approximation. So we'll use this instead

$$P(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(n-\bar{n})^2}{2\sigma^2}\right], \quad \bar{n} = Nq_1, \quad \sigma^2 = q_0 q_1 N$$

Consider $n = \bar{n}(1+\delta)$

$$\bar{n} - n = \bar{n}\delta = \delta q_1 N$$

$$\frac{\bar{n}-n}{\sigma} = \frac{\delta q_1 N}{\sqrt{q_1 q_0 N}} = \delta \sqrt{\frac{q_1}{q_0}} \sqrt{N}$$

$$P(\bar{n}(1+\delta)) \sim \exp\left[-\delta^2 \frac{q_1}{2q_0} N\right]$$

For any arbitrarily small but finite δ, this decays exponentially with N. So too does the prob. that $n \geq \bar{n}(1+\delta)$ since

$$\sum_{j=\bar{n}(1+\delta)}^{N} P(j) \leq \left(N - \bar{n}(1+\delta)\right) P(n)$$

So if we encode all $n < \bar{n}(1+\delta)$, we'll have an arbitrarily small ε.

The number of states that this corresponds to is

$$\sum_{j=0}^{\bar{n}(1+\delta)-1} \binom{N}{j} \leq (\bar{n}(1+\delta)-1) \binom{N}{\bar{n}(1+\delta)} \leq N \binom{N}{\bar{n}(1+\delta)}$$

Let's take the far right to be k.

The number of qubits required to encode this is then

$$K_N = \log_2 k = \log_2 N + \log_2 \binom{N}{\bar{n}(1+\delta)}$$

Note that $\bar{n}(1+\delta) = N q_1 (1+\delta)$

Using the relation in lecture then gives

$$\log_2 \binom{N}{\bar{n}(1+\delta)} \leq N\, H(q_1(1+\delta))$$

So $\dfrac{K_N}{N} \leq H(q_1(1+\delta)) \quad N \to \infty$

Also

$$H(q_1(1+\delta)) = H(q_1) + O(\delta)$$

So there you go