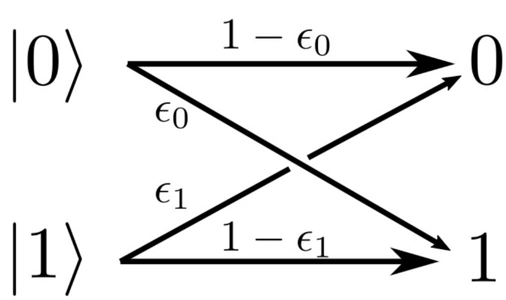
Exercise 3

Due November 25 before noon

1 Readout errors and stochastic matrices

At the end of a quantum computation, the state of the qubits is measured. The act of measurement is a classical communication problem — the computer is sending messages to the user. Let's say that at the end of the computation, the probability that the qubit is in state $|1\rangle$ is p, while the probability that the qubit is in $|0\rangle$ is $1-p$. A noisy measurement device is then used to make an inference — "0" or "1" — about the state of the qubit, with errors ϵ_0 and ϵ_1 described by an asymmetric bitflip channel:



(a) Write down the matrix S that relates the vector of pre-measurement probabilities, $P = (1-p, p)^\top$, to the vector of post-measurement probabilities, $P' = (1-p', p')^\top$. The matrix should satisfy $P' = SP$. (**2 marks**)

Reading off the diagram, we have

$$\begin{pmatrix} 1-p' \\ p' \end{pmatrix} = \underbrace{\begin{pmatrix} 1-\epsilon_0 & \epsilon_1 \\ \epsilon_0 & 1-\epsilon_1 \end{pmatrix}}_{=S} \begin{pmatrix} 1-p \\ p \end{pmatrix} \tag{1}$$

.

(b) The measurement fidelity F gives the probability of correctly identifying the qubit state in a given measurement. Consider the case of equal priors, $p = 1/2$. What is F in terms of $\operatorname{Tr} S$? Hint: The fidelity can be computed as $F = P(\text{"0"}, 0) + P(\text{"1"}, 1)$, where $P(\text{"}j\text{"}, j)$ is the *joint* probability of the qubit being in state $|j\rangle$ and of correctly inferring the outcome "j". (**2 marks**)

From Bayes' Rule, we have

$$F = P(\text{"0"}, 0) + P(\text{"1"}, 1) \tag{2}$$

$$= P(\text{"0"}|0)P(0) + P(\text{"1"}|1)P(1), \tag{3}$$

where $P(\text{“}j\text{”}|j)$ is the conditional probability of inferring j *given* a qubit in state $|j\rangle$. In terms of the notation established previously, the quantities appearing above are given by

$$P(\text{“0”}|0) = 1 - \epsilon_0, \tag{4}$$
$$P(\text{“1”}|1) = 1 - \epsilon_1, \tag{5}$$
$$P(0) = 1 - p, \tag{6}$$
$$P(1) = p. \tag{7}$$

For equal priors, $p = 1/2$, it then follows that

$$F = \frac{1}{2}\left[(1-\epsilon_0) + (1-\epsilon_1)\right] = \frac{1}{2}\mathrm{Tr}\, S. \tag{8}$$

(c) A matrix M is stochastic if (1) $M_{ij} \geq 0\ \forall i, j$ and (2) $\sum_i M_{ij} = 1\ \forall j$. If S^{-1} is *not* stochastic, then the channel is irreversible and information has been lost. Compute S^{-1} for $\epsilon_0 = \epsilon_1 = \epsilon$. Find all values of ϵ for which S^{-1} is a stochastic matrix. Explain why your result makes sense. (**4 marks**)

The inverse matrix S^{-1} exists for $\epsilon \neq 1/2$ and is given by

$$S^{-1} = \frac{1}{1-2\epsilon}\begin{pmatrix} 1-\epsilon & -\epsilon \\ -\epsilon & 1-\epsilon \end{pmatrix}. \tag{9}$$

You can immediately see for yourself that the columns add up to 1: $\sum_i [S^{-1}]_{ij} = 1 \forall j$. For the matrix to be stochastic, we also require that all its elements be non-negative. Focusing on the diagonal elements first, we have the requirement that

$$\frac{1-\epsilon}{1-2\epsilon} \geq 0. \tag{10}$$

This is satisfied for $\epsilon = 1$ and for $\epsilon \leq 1/2$. Meanwhile, from the off-diagonal elements, we have the requirement that

$$\frac{-\epsilon}{1-\epsilon} \geq 0. \tag{11}$$

This is satisfied for $\epsilon = 0$ and $\epsilon \geq 1/2$. The inverse is therefore stochastic only for $\epsilon = 1$, in which case

$$S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

and for $\epsilon = 0$, in which case

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

There are therefore two scenarios in which the bitflip channel does not lead to a loss of information: (1) the scenario where there are no readout errors at all ($\epsilon = 0$), or (2) the scenario where the inferred state is always opposite the actual qubit state ($\epsilon = 1$). In the latter case, you can undo the errors by simply flipping the inferred bit value. All other values $\epsilon \neq 0, 1$ involve a probabilistic mapping and some level of uncertainty in the measured outcome, amounting to a loss of information.

2 Control power in quantum teleportation

Let's consider a three-party quantum teleportation scheme involving Alice, Bob, and Charlie. Alice has a single-qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ that she wants to send to Bob. Alice, Bob, and Charlie share a three-qubit GHZ state given by $|\Psi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. Charlie is an agent of chaos who may or may not cooperate. Let's see whether Alice and Bob can still benefit from shared entanglement without Charlie's help.

(a) Alice performs a Bell measurement on $|\psi\rangle$ and her portion of the shared GHZ state. She obtains one of four measurements, given by

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{14}$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{15}$$

Calculate the post-measurement state of Bob and Charlie's qubits conditioned on the outcome of Alice's Bell measurement. (Make sure this state is properly normalized.) Record your answers in the following table (**2 marks**):

Prior to the Bell measurement, the state of all four qubits is given by

$$|\psi\rangle |\Psi\rangle = \frac{1}{\sqrt{2}} \left[a |00\rangle |00\rangle + a |01\rangle |11\rangle + b |10\rangle |00\rangle + b |11\rangle |11\rangle\right]. \tag{16}$$

It then helps to note that

$$|00\rangle = \frac{1}{\sqrt{2}}(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle), \tag{17}$$

$$|11\rangle = \frac{1}{\sqrt{2}}(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle), \tag{18}$$

$$|01\rangle = \frac{1}{\sqrt{2}}(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle), \tag{19}$$

$$|10\rangle = \frac{1}{\sqrt{2}}(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle). \tag{20}$$

This lets us read the post-measurement states off of Eq. (16):

Alice's outcome	State of Bob and Charlie's qubits
$\lvert\Phi^+\rangle$	$a \lvert 00\rangle + b \lvert 11\rangle$
$\lvert\Phi^-\rangle$	$a \lvert 00\rangle - b \lvert 11\rangle$
$\lvert\Psi^+\rangle$	$a \lvert 11\rangle + b \lvert 00\rangle$
$\lvert\Psi^-\rangle$	$a \lvert 11\rangle - b \lvert 00\rangle$

(b) Charlie is in a good mood and decides to help out. He now performs an X-basis measurement of his qubit and communicates the outcome to Bob. Building on your answer from (a), write down the state of Bob's qubit following Alice's Bell measurement and Charlie's X-basis measurement. Identify the correction operator Bob needs to apply to recover the original state $|\psi\rangle$. Assume a basis ordering in which $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, and record your answers in the table below (**4 marks**).

Similar to the last question, it helps to write the computational basis states $|0\rangle$ and $|1\rangle$ in terms of the eigenstates $|\pm\rangle$ of the measured operator:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \tag{21}$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \tag{22}$$

We can use this to fill in the table as follows.

Alice's outcome	Charlie's outcome	State of Bob's qubit	Correction
$\|\Phi^+\rangle$	$\|+\rangle$	$a\|0\rangle + b\|1\rangle$	$\mathbb{1}$
$\|\Phi^+\rangle$	$\|-\rangle$	$a\|0\rangle - b\|1\rangle$	Z
$\|\Phi^-\rangle$	$\|+\rangle$	$a\|0\rangle - b\|1\rangle$	Z
$\|\Phi^-\rangle$	$\|-\rangle$	$a\|0\rangle + b\|1\rangle$	$\mathbb{1}$
$\|\Psi^+\rangle$	$\|+\rangle$	$a\|1\rangle + b\|0\rangle$	X
$\|\Psi^+\rangle$	$\|-\rangle$	$a\|1\rangle - b\|0\rangle$	ZX
$\|\Psi^-\rangle$	$\|+\rangle$	$a\|1\rangle - b\|0\rangle$	ZX
$\|\Psi^-\rangle$	$\|-\rangle$	$a\|1\rangle + b\|0\rangle$	X

Note that *with* Charlie's cooperation, Bob can recover Alice's state with perfect fidelity.

(c) Charlie is in a bad mood and wants no part in Alice and Bob's shenanigans. He measures his qubit but does not record the outcome. Without Charlie's help, Bob does his best to reconstruct Alice's original state $|\psi\rangle$: Having received the outcome of Alice's Bell measurement, he applies the following correction operator to his qubit conditioned on Alice's outcome:

$$\left|\Phi^+\right\rangle \to \mathbb{1}, \tag{23}$$

$$\left|\Phi^-\right\rangle \to Z, \tag{24}$$

$$\left|\Psi^+\right\rangle \to X, \tag{25}$$

$$\left|\Psi^-\right\rangle \to ZX. \tag{26}$$

Building on your results from (a), apply the correction given above, then trace over Charlie's qubit to model Charlie's lack of cooperation. In other words, evaluate the state ρ_{B} of Bob's qubit as

$$\rho_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{C}}(\rho_{\mathrm{BC}}), \tag{27}$$

where ρ_{BC} is the state of Bob and Charlie's qubits with the appropriate correction operator already applied. Note that tracing over Charlie's qubit amounts to discarding the outcome of Charlie's measurement. (**3 marks**)

With the appropriate correction operator applied, the state of Bob and Charlie's qubits is given by $a|00\rangle + b|11\rangle$ following a measurement of $|\Phi^{\pm}\rangle$, and by $a|01\rangle + b|10\rangle$ following a measurement of $|\Psi^{\pm}\rangle$. In all cases, the state ρ_{B} of Bob's qubit obtained by tracing over Charlie's qubit is given by

$$\rho_{\mathrm{B}} = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1| . \tag{28}$$

(d) For an initial state $|\psi\rangle$, $F_\psi = \langle\psi|\rho_{\mathrm{B}}|\psi\rangle$ gives the fidelity of the teleportation achieved without Charlie's help. Average F_ψ over the single-qubit Haar measure $d\psi$ to obtain the state-averaged fidelity $\bar{F}$ of the teleportation protocol:

$$\bar{F} = \int d\psi \, \langle\psi|\rho_{\mathrm{B}}|\psi\rangle. \tag{29}$$

(**3 marks**) Hint: This average can be performed as an average over the polar and azimuthal coordinates θ and ϕ as

$$\int d\psi\,(\cdots) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\,(\cdots). \tag{30}$$

The factor of 4π in the denominator accounts for the 4π solid angle of the Bloch sphere. You may want to express a and b in terms of θ and ϕ.

The highest attainable fidelity for a "classical" teleportation protocol is $\bar{F} = 2/3$. Such a strategy involves Alice making a measurement of her qubit (which is *not* entangled with Bob's qubit), sending the outcome to Bob over a classical communication channel, and having Bob perform operations on another qubit in an attempt to reconstruct Alice's original state. If Charlie withholds his measurement outcome, does quantum teleportation with the GHZ state provide an advantage over classical resources? (**1 mark**)

In terms of the polar and azimuthal angles on the Bloch sphere, the coefficients a and b used to parametrize the initial state can be written as

$$a = \cos(\theta/2), \tag{31}$$

$$b = e^{i\phi}\sin(\theta/2). \tag{32}$$

Since $\langle\psi|\rho_{\mathrm{B}}|\psi\rangle = |a|^4 + |b|^4$, we then have

$$\bar{F} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \left(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}\right) = \frac{2}{3}. \tag{33}$$

This is exactly equal to the classical bound. Therefore, without Charlie's cooperation, Alice and Bob do NOT derive any benefits from their shared entanglement. The protocol sketched in this question is an example of "controlled" quantum teleportation [Karlsson and Bourennane, PRA (1998)].

(e) Let's say we wanted to implement such a teleportation scheme experimentally, and to measure the state-averaged fidelity $\bar{F}$. Having the initial state $|\psi\rangle$ parameterized by a continuum of angles (θ, ϕ) would make evaluating $\bar{F}$ somewhat inconvenient in such a scenario. (What θ, ϕ do we choose? How finely do we need to sample the Bloch sphere, and would a finite mesh even reproduce the statistics of the Haar measure $d\psi$?) Express $\langle\psi|\rho_{\mathrm{B}}|\psi\rangle$ as a polynomial $p(\boldsymbol{x})$ in two variables a and b, i.e. $\boldsymbol{x} = (a, b)^\top$, and compute (**2 marks**)

$$\frac{1}{|X|}\sum_{\boldsymbol{x}\in X} p(\boldsymbol{x}). \tag{34}$$

Here, $|X| = 6$ is the cardinality of the set X, i.e. the number of elements in X, while X itself is given by

$$X = \left\{(1,0), (0,1), \frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,i), \frac{1}{\sqrt{2}}(1,-i)\right\}. \tag{35}$$

Does this quantity look familiar? If you were an experimentalist, what initial states $|\psi\rangle$ would you prepare in order measure the average fidelity of your teleportation protocol *without* averaging over all $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$? (**1 mark**)

We've already written the state-conditioned fidelity $\langle\psi|\rho_{\mathrm{B}}|\psi\rangle$ as a polynomial (see previous question):

$$p(a, b) = |a|^4 + |b|^4. \tag{36}$$

The sum can then be evaluated by plugging each element of X into this polynomial and summing to obtain

$$\frac{1}{|X|}\sum_{\boldsymbol{x}\in X} p(\boldsymbol{x}) = \frac{1}{6}\left[1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right] = \frac{2}{3}. \tag{37}$$

This is exactly equal to the value obtained from explicitly averaging over the Haar measure. The set X is an example of a complex projective 2-design — a finite set of states which, when averaged over, reproduce the statistics of the Haar measure. We could therefore measure the fidelity of this controlled quantum teleportation protocol by preparing the six eigenstates of the Pauli matrices X, Y, and Z (whose coefficients in the computational basis correspond to the elements of X above).