1 For each vector $|\psi_k\rangle$ we need to find a vector $|\tilde{\psi}_k\rangle$ that is not orthogonal to $|\psi_k\rangle$, but is orthogonal to all the others.

Given a set of linearly independent vectors, we can use the Gram-Schmidt process to find a set of orthogonal vectors that span the same space.

$\mathbb{P}_j = \mathbb{1} - |\phi_j\rangle\langle\phi_j|$ is the projector onto all that is orthogonal to $|\phi_j\rangle$

$$|\phi_1\rangle = |\psi_1\rangle$$

$$|\phi_2\rangle = \mathbb{P}_1 |\psi_2\rangle$$

$$|\phi_3\rangle = \mathbb{P}_2 \mathbb{P}_1 |\psi_3\rangle$$

...

If we do this for the set $\{|\psi_j\rangle\} \setminus |\psi_k\rangle$ for each we can define the state

$$|\tilde{\psi}_k\rangle = \prod_{j \neq k} \mathbb{P}_j |\psi_j\rangle$$

A POVM can then be defined as

$$E_k = |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k| \quad \forall k \leq m$$

$$E_{m+1} = \mathbb{1} - \sum_{k=1}^{m} E_k$$

2

By definition

$$H(XY) = -\sum_{x,y} p(x,y) \log p(x,y)$$

By definition, and using $\sum_{y} p(x,y) = p(x)$, etc

$$H(X) + H(Y) = -\sum_{x,y} p(x,y)\left[\log p(x) + \log p(y)\right]$$

$$= -\sum_{x,y} p(x,y) \log p(x)\,p(y)$$

So

$$H(X) + H(Y) - H(XY) = -\sum_{x,y} p(x,y) \log \frac{p(x)\,p(y)}{p(x,y)}$$

In N+C Eq.s 11.24 and 11.25 show that

$$\sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)} \leq 0$$

$\therefore \quad H(X) + H(Y) - H(XY) \geq 0$

$\therefore \quad H(XY) \leq H(X) + H(Y)$

3.

a) $\rho = \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & \ddots \end{pmatrix} \quad \sigma = \begin{pmatrix} q_1 & & \\ & q_2 & \\ & & \ddots \end{pmatrix}$

$$\rho \otimes \sigma = \begin{pmatrix} p_1 q_1 & & & & & \\ & p_1 q_2 & & & & \\ & & \ddots & & & \\ & & & p_2 q_1 & & \\ & & & & p_2 q_2 & \\ & & & & & \ddots \end{pmatrix}$$

$S(\rho) = -\sum_j p_j \log p_j, \quad S(\sigma) = -\sum_k q_k \log q_k$

$$S(\rho \otimes \sigma) = -\sum_{jk} p_j q_k \log p_j q_k$$

$$= -\sum_{jk} p_j q_k (\log p_j + \log q_k)$$

$$= -\sum_j p_j \log p_j - \sum_k q_k \log q_k$$

$$= S(\rho) + S(\sigma)$$

b) $|\psi\rangle\langle\psi| = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \\ & & & \ddots \end{pmatrix}$

$\therefore \quad S(|\psi\rangle\langle\psi|) = 1 \times \log 1 + (d-1)\, 0 \times \log 0$

where d is the system dimension

We use the convention $0 \times \log 0 = 0$

$\therefore \quad S(|\psi\rangle\langle\psi|) = 0$