Planar Code as a Topologically Ordered System

So far we have considered the planar code only as an error correcting code

Now we will define a Hamiltonian for it, and start to think of it as a condensed matter system

Then we can ask

What kind of order is present in the gs?

What about finite temperature?

What happens in the presence of local perturbations?

Most straightforward Hamiltonian is one that energetically penalizses anyons

$$H = -J\sum_v A_v - J'\sum_p B_p$$

The ground state space is the stabilizer space

Eigenstates are states of e and m anyons

These are 'quasiparticles' of the system, localized excitations

Energy for N_e e's and N_m m's is

$$2JN_e + 2J'N_m$$

Note that this has no position dependence

Apart from hardcore repulsion, anyons do not interact (though we could write down Hamiltonians where they do)

Topological Order

What kind of order is in the gs? FM? AFM? Spin glass? Something else?

Turns out it has 'topological order'

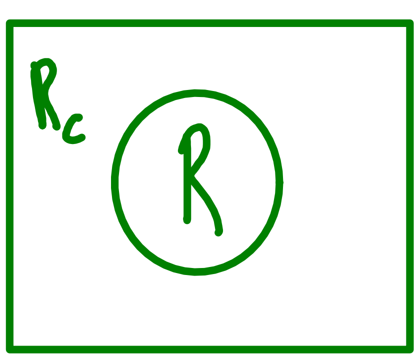
This cannot be detected by a local order parameter (like ferromagnetism)

Instead, to detect it we can use entropies

The area law for ground states of gapped systems of interacting spins is:

Consider a region R

$\rho_R = \mathrm{tr}_{R_c} |gs\rangle\langle gs|$



The entropy of this will take the form

$$S(\rho_R) = \alpha L_R - n_c \gamma + \dots$$

boundary term

WTF!?

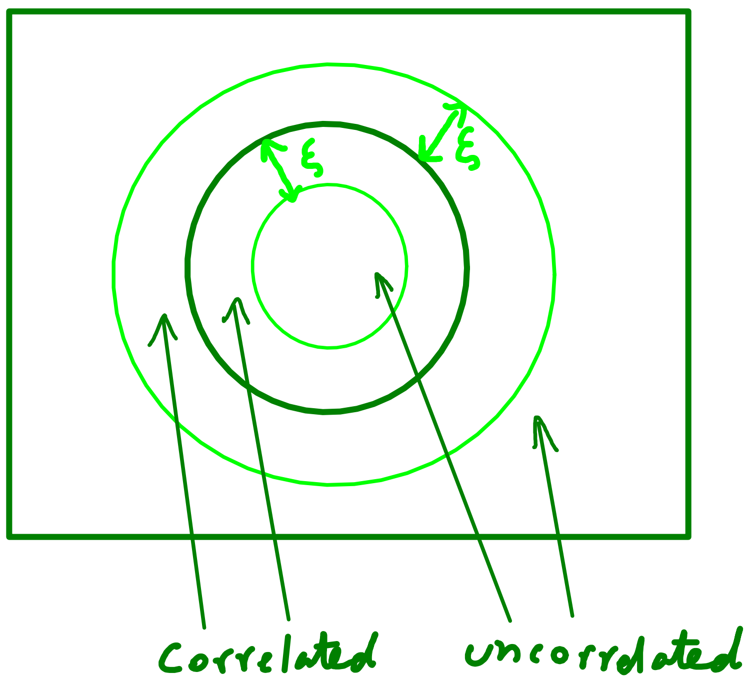
terms that disappear for large R

Why does this law hold? $S(\rho_R) = \alpha L_R - n_c \gamma + \ldots$

The entropy measures correlations between R and the rest

$$S(\rho_{R \cup R_c}) = S(|gs\rangle) = 0, \quad S(\rho_{R_c}) = S(\rho_R) \quad \therefore \; I(R;R_c) = 2S(\rho_R)$$

Because the system is gapped, correlations are short-ranged. Only spins either side of the boundary are correlated



This gives rise to the first term

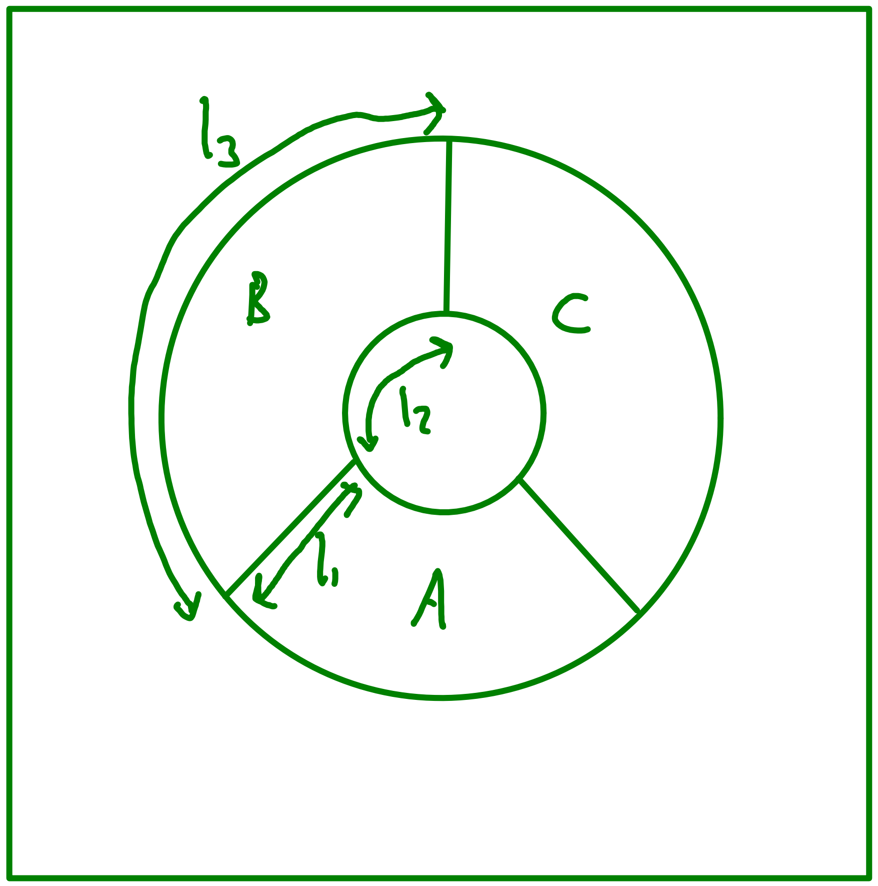
entropy due to correlations over boundary $\sim$ surface area of region R

For the next, n_c is the number of disconnected boundary curves, and γ does not depend on L_R

So the whole term depends only on the topology of R

γ is non-zero only for topologically ordered systems

Consider regions A, B, C, $A\cup B$, $A\cup C$, $B\cup C$, $A\cup B\cup C$ (can be different sized)



Clearly

$$n_A = n_B = n_C = n_{AB} = n_{AC} = n_{BC} = 1$$

$$n_{ABC} = 2$$

$$L_A = L_B = L_C = 2l_1 + l_2 + l_3$$

$$L_{AB} = L_{AC} = L_{BC} = 2l_1 + 2l_2 + 2l_3$$

$$L_{ABC} = 3l_2 + 3l_3$$

Using $S_R = \alpha L_R - n_R \gamma$

We find $I_{A;B} = I_{A;C} = I_{B;C} = 2(2l_1 + l_2 + l_3 - \gamma) - (2l_1 + 2l_2 + 2l_3 - \gamma) = 2l_1 - \gamma$

$$I_{A;BC} = (2l_1 + l_2 + l_3 - \gamma) + (2l_1 + 2l_2 + 2l_3 - \gamma) - (3l_2 + 3l_3 - 2\gamma) = 4l_1$$

$$\therefore \Gamma = I_{A;BC} - I_{A;B} - I_{A;C} = 2\gamma$$

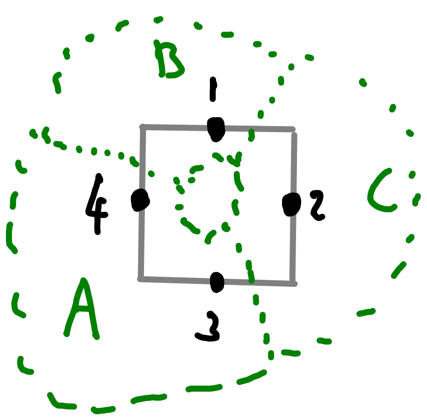
Γ is the information A shares with $B\cup C$, but not B or C alone

It measures loop correlations

If we can find loop correlations for arbitrarily long loops, the system is topological ordered

Is this true of the planar code ground state?

Lets consider a single plaquette



Clearly ρ_R is an equally weighted mixture of all σ_z basis states $|ijkl\rangle$ for which

$i,j,k,l \in \{0,1\}$ and $i+j+k+l \mod 2 = 0$

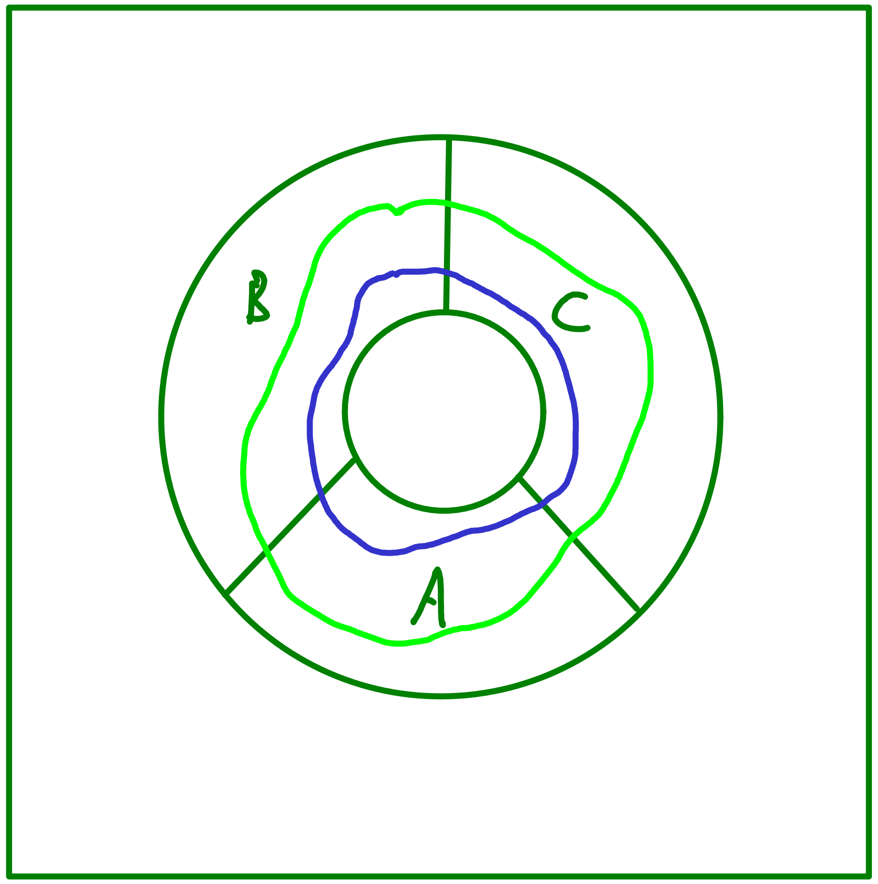
This implies $i+j \mod 2 = k+l \mod 2$, so $I_{A;BC} = 1$

But there are no correlations between A and B or A and C

$\rho_{134} = \rho_{234} \propto \mathbb{1} \quad \therefore \quad I_{A;B} = I_{A;C} = 0$

$\therefore \quad I_{A;BC} - I_{A;B} - I_{A;C} = 1 - 0 - 0 = 1 > 0$

So single plaquettes have loop correlations at least!
But what about arbitrarily large loops?



On an annulus we can define a large loop operator, which is a product of A_v for all vertices it encloses: L_v

Can also do one for B_p : L_p

These are two independent loop operators

gs is +1 eigenstate of both

Use $L_{p|B}$ to denote the part of L_p in region A, etc

Note that the Shannon entropies of 0's and 1's around L_p and +'s and -'s around L_v satsify

$$H(L_{p|A}),\ H(L_{p|B}),\ H(L_{p|C}),\ H(L_{v|A}),\ \ldots > 0$$

Since the gs is a +1 eigenstate of both operators, each must contribute at least one bit of information to $I_{A;BC}$, but not to $I_{A;B}$ or $I_{A;C}$

So the planar code has topological order! $\Gamma = 2 - 0 - 0 = 2 > 0$

Finite Temperature

This is for the gs, but does the TO survive at finite T?

Boltzmann dist. gives (unnormalized) probability $e^{-\beta \varepsilon_j}$ for each eigenstate of H with energy ε_j at temp. $T = 1/\beta$

Normalization constant is the partition function

$$Z = \sum_j e^{-\beta \varepsilon_j}$$

For a quantum system this takes the form

$$Z = \mathrm{tr}\left(e^{-\beta H}\right)$$

The quantum state with this distribution is known as the Gibbs state

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}\left(e^{-\beta H}\right)}$$

To determine whether the planar code has TO at T>0 we calculate Γ using this state

To calculate this, note that the planar code H acts on each vertex and plaquette independently

$$H = -J\sum_{v} A_v - J'\sum_{p} B_p$$

This allows us to factorize the partition function and Gibbs state

stabilizers commute $\therefore$ $$e^{-\beta H} = \prod_{v} e^{\beta J A_v} \prod_{p} e^{\beta J' B_p}$$

Since these exponentials are a product of Paulis

$$Z = \mathrm{tr}(e^{-\beta H}) = \prod_{v} \mathrm{tr}(e^{\beta J A_v}) \prod_{p} \mathrm{tr}(e^{\beta J' B_p})$$

$$\rho = \frac{e^{-\beta H}}{Z} = \prod_{v} \frac{e^{\beta J A_v}}{\mathrm{tr}(e^{-\beta J A_v})} \prod_{p} \frac{e^{\beta J' B_p}}{\mathrm{tr}(e^{-\beta J B_p})} = \prod_{v} \rho_v \prod_{p} \rho_p$$

Expressing the stabilizers as $A_v = \mathbb{P}_v^1 - \mathbb{P}_v^e \quad B_p = \mathbb{P}_p^1 - \mathbb{P}_p^m$

$$\rho_v = \frac{e^{\beta J A_v}}{\mathrm{tr}(e^{-\beta J A_v})} = \frac{e^{\beta J}\mathbb{P}_v^1 + e^{-\beta J}\mathbb{P}_v^e}{\mathrm{tr}(e^{\beta J}\mathbb{P}_v^1 + e^{-\beta J}\mathbb{P}_v^e)} = p_v^1 \frac{\mathbb{P}_v^1}{8} + p_v^e \frac{\mathbb{P}_v^e}{8}$$

The probabilities that the vertices and plaquettes hold anyons is then

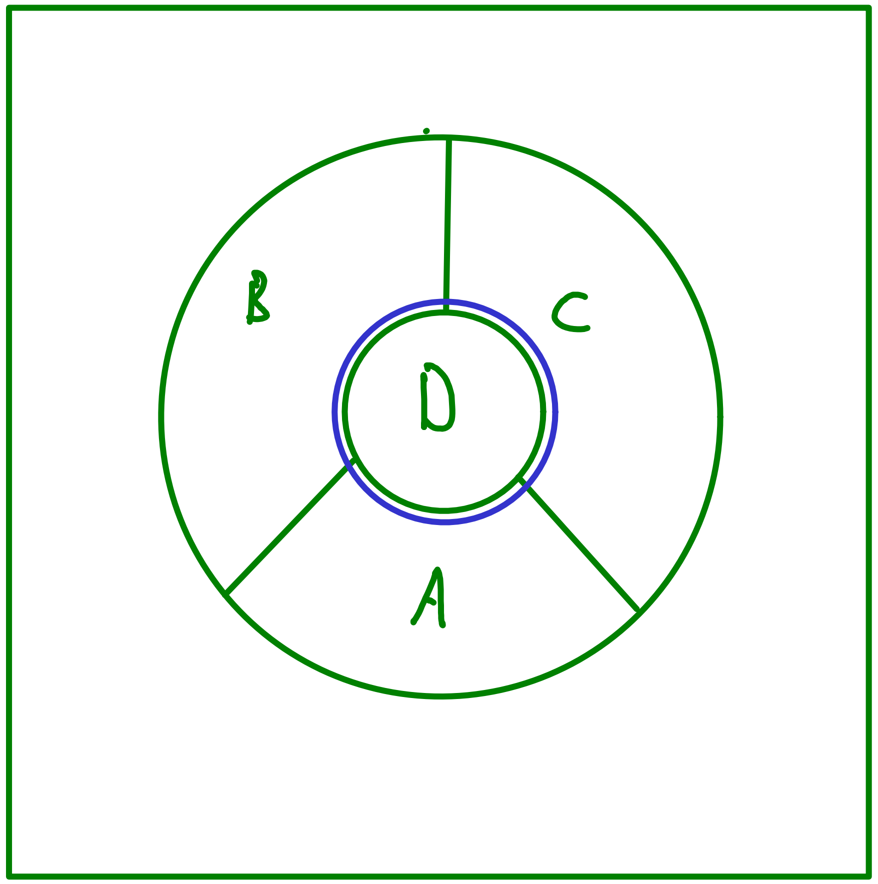
$$p_v^e = \frac{e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} = (1 + e^{2\beta J})^{-1} \quad \text{and similarly} \quad p_p^m = (1 + e^{2\beta J'})^{-1}$$

The thermal state has a random configuration of anyons, occuring with these i.i.d. probabilities.

Does the TO survive?

Consider the loop operator made out of all B_p with full or partial support on region D

$$L_p = \prod_{p \in D} B_p$$



Has support on $A \cup B \cup C$

Has eigenvalues +1 when there is an even number of m's in $A \cup B \cup C$ and -1 when there is an odd number

This means

Parity of # 1's around loop = parity of m's enclosed by loop

Same for L_v m e's and -'s

When parity of anyons becomes random, so does parity of 1's and -'s. Without a definite (or biased) parity $I_{A;BC} = 0$

As such $\Gamma = 0$, and so we lose TO at finite T

Perturbed Planar Code

- Let's return to $T=0$
- What happens if the Hamiltonian is perturbed? Does T_0 survive?
- Example

$$H=-J\sum_V A_v - J'\sum_p B_p - h\sum_i \sigma_x^i$$

- Since $[A_v, \sigma_x^i]=0 \quad \forall\, v, i$, the vertex terms aren't really important. So instead we consider simply

$$H=-J'\sum_p B_p - h\sum_i \sigma_x^i$$

- We can think of each plaquette as a pseudo spin, or pseudo qubit

$$\text{No } m \text{ anyon} \equiv |+\rangle$$

$$\text{an } m \text{ anyon} \equiv |-\rangle$$

$$\therefore \quad B_p \equiv \sigma_x^p$$

- Because σ_x^i anticommutes with the 2 plaquettes it touches

$$\sigma_x^i \equiv \sigma_z^p \sigma_z^{p'}$$

· So
$$H = -J' \sum_p B_p - h \sum_i \sigma_x^i \equiv -J \sum_j \sigma_x^j - h \sum_{\langle j,k \rangle} \sigma_z^j \sigma_z^k = H_{TFIM}$$

· More explicitly, these two Hamiltonians are unitarily equivalent

· The Hamiltonian H_{TFIM} is that of the transverse field Ising model

$h \ll J \Rightarrow$ Spin polarized phase $\equiv$ TO phase

$h \gg J \Rightarrow$ FM phase $\equiv$ Non-topological phase (spin polarized)

Phase transition at $\frac{h}{J} = 0.328$

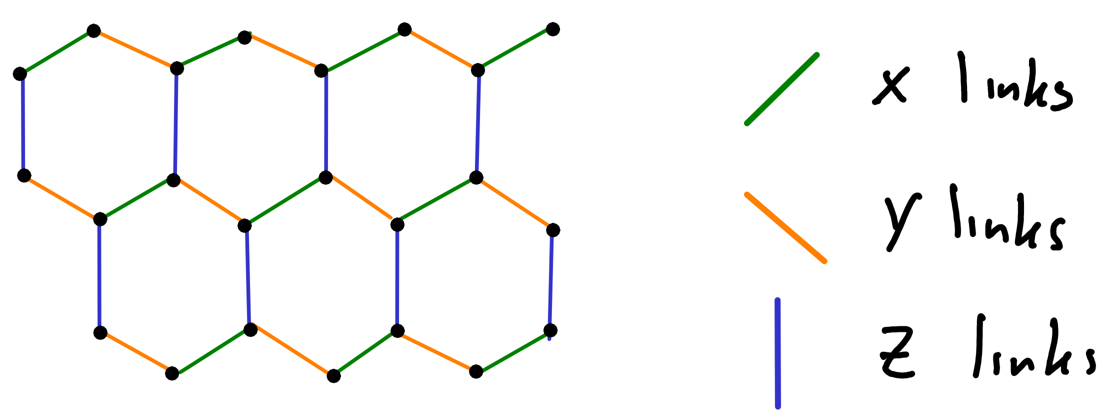
· So topological order persists up to a critical perturbation strength

· True for any type of local perturbation, for any gapped topological model (though critical value will differ)

TO Phase = always stable against local perturbations
Not always stable against temperature

Honeycomb lattice Model

- Hamiltonians with 4-body interactions are hardly realistic
- 2-body interactions are what nature gives us
- Can the planar code Hamiltonian be engineered using these?
- Consider a model defined on a honeycomb lattice (spins on vertices)



$$H = -J_x \sum_{x\text{-links}} \sigma_x^i \sigma_x^j - J_y \sum_{y\text{-links}} \sigma_y^i \sigma_y^j - J_z \sum_{z\text{-links}} \sigma_z^i \sigma_z^j$$

- Clearly terms do not commute $\therefore$ hard to solve
- Consider the case of $J_z \gg J_x, J_y$. The x and y links are perturbations on the z
- States $|01\rangle$ and $|10\rangle$ on z links are highly suppressed, and can be ignored

· Each z link l ∴ can be described by a single effective qubit

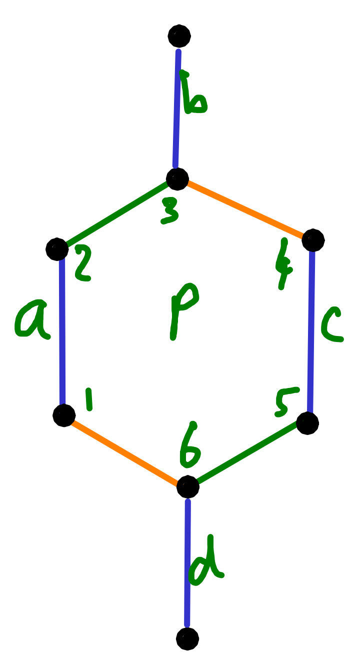
$|0\rangle_l = |00\rangle \qquad |1\rangle_l = |11\rangle$

$$\left.\begin{aligned} &\sigma_z^i \sigma_z^j \equiv \mathbb{1} \\ &\sigma_x^i \sigma_x^j \equiv \sigma_x^l \\ &\sigma_z^i \equiv \sigma_z^j \equiv \sigma_z^l \\ &\sigma_x^i \sigma_y^j \equiv \sigma_y^i \sigma_x^j = -i\,\sigma_y^l \\ &\sigma_x^i, \sigma_x^j, \sigma_y^i, \sigma_y^j \text{ are suppressed} \end{aligned}\right\} \quad l = (i,j)$$

· What is the effective Hamiltonian on these?

· Hand waving perturbation theory: find the minimum products of perturbations such that the products commute with the z links

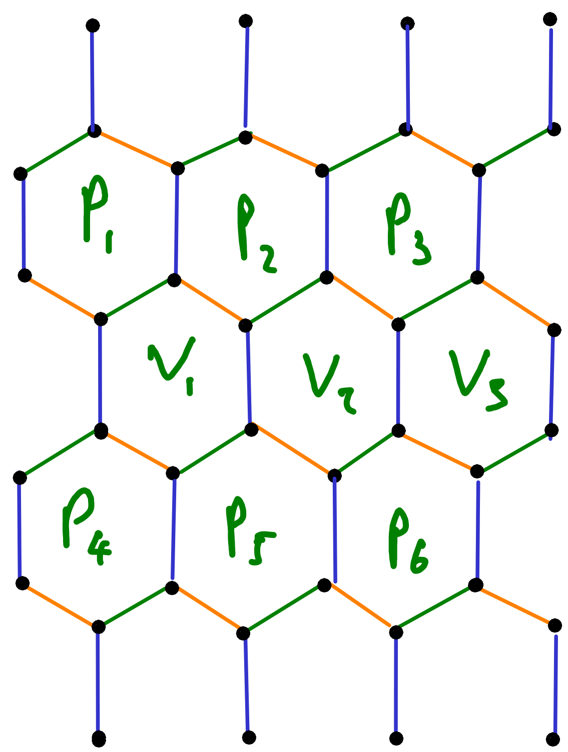
In this case



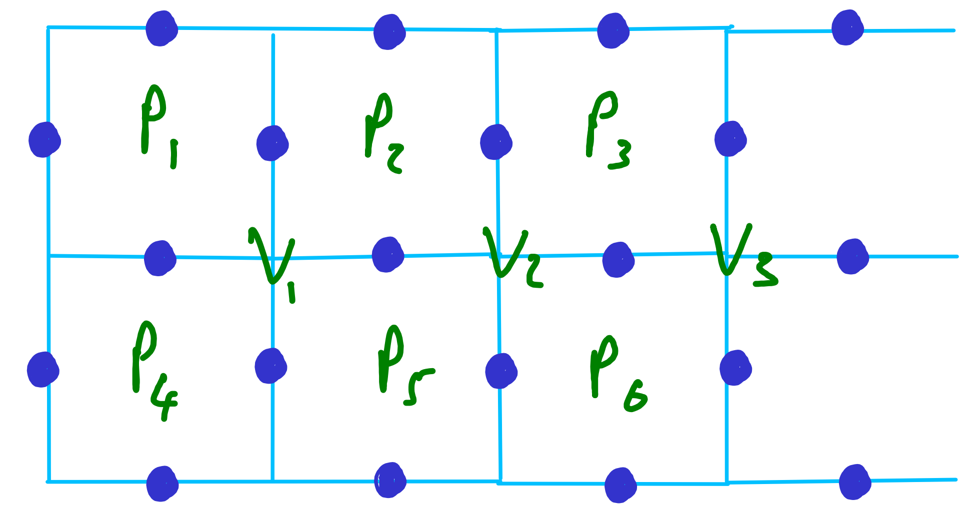
$$Q_p = (\sigma_x^2 \sigma_x^3)(\sigma_y^3 \sigma_y^4)(\sigma_x^5 \sigma_x^6)(\sigma_y^6 \sigma_y^1)$$
$$= -\sigma_x^2 \sigma_z^3 \sigma_y^4 \sigma_x^5 \sigma_z^6 \sigma_y^1$$
$$= \sigma_y^{l_a} \sigma_z^{l_b} \sigma_y^{l_c} \sigma_z^{l_d}$$

$$H_{eff} = -J_{eff} \sum_p Q_p \qquad J_{eff} = \frac{J_x^2 J_y^2}{16 |J_z^3|}$$

· The effective spins of the z links form a square lattice



$\equiv$



$$H_{eff} = \sum_{p,v} \sigma_y^{left} \sigma_z^{top} \sigma_y^{right} \sigma_z^{bottom}$$

· This is unitarily equivalent to the standard planar code Hamiltonian

Simply apply the phase gate $P^\dagger$ to qubits on vertical links

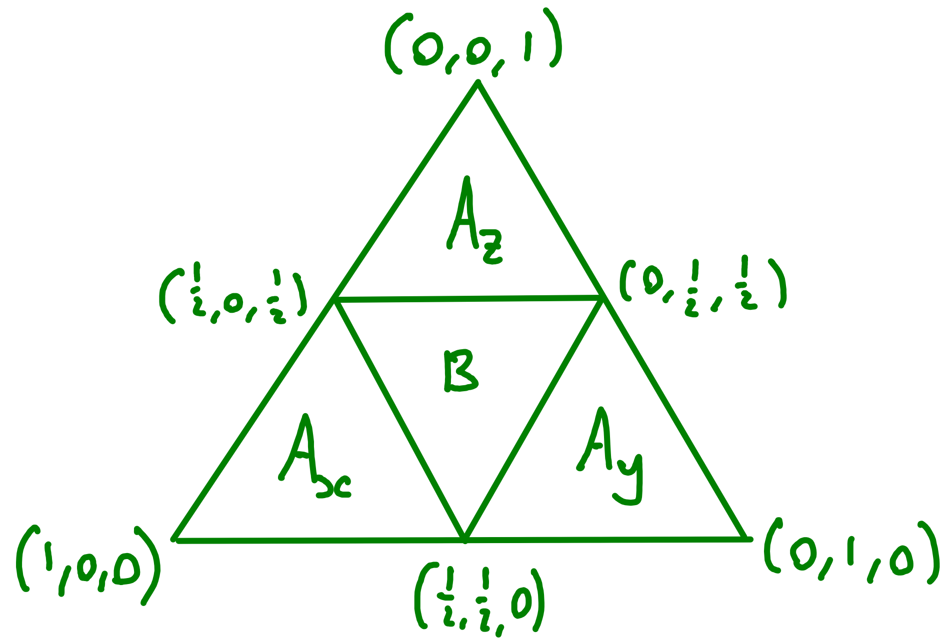
$$P^\dagger \sigma_y P^\dagger = \sigma_x \qquad U = \prod_{\substack{j \text{ on vertical} \\ \text{links}}} P_j^\dagger \qquad U H_{eff} U^\dagger = -J_{eff}\left(\sum_v A_v + \sum_p B_p\right)$$

· So we effectively have the planar code Hamiltonian, using 2 body terms

· Same would happen if we took $J_x \gg J_y, J_z$

$J_y \gg J_x, J_z$

· Phase diagram is



where co-ordinates are $(J_x, J_y, J_z) \geq 0$

· The phases A_z, A_x, A_y are equivalent, just defined on the effective spins of different links

· B is a place where lots of strange things can happen