Stabilizer formalism

The repetition code, surface code, Shor code, color code and many others use the stabilizer formalism

Suppose we wish to store k logical qubits in n physical ones

We select n-k mutually commuting operators from the set of all n qubit tensor products of Paulis

$$\sigma_{a_1}^{1}\,\sigma_{a_2}^{2}\,\sigma_{a_3}^{3}\cdots\sigma_{a_n}^{n}, \qquad a_j \in \{0, x, y, z\}, \qquad \sigma_0 = \mathbb{1}$$

These must be independent (none can be expressed as a product of others)

These are our check operators $\{S_j\}$, which we call stabilizers

Since they are products of Paulis, they too are observables with eigenvalues +1/-1. So they define measurements corresponding to the projectors

$$\mathbb{P}_+^j = \frac{1}{2}(\mathbb{1} + S_j), \qquad \mathbb{P}_-^j = \frac{1}{2}(\mathbb{1} - S_j)$$

The results of these are called the syndrome

Consider the states which are within the +1 eigenspace of all stabilizers. We call the space spanned by these the stabilizer space

Stabilizer space: $S_j |\psi\rangle = |\psi\rangle \quad \forall j$

Since there are n-k independent stabilizers on an n qubit space, the stabilizer space is 2^k dimensional. It is within this space that we store the k logical qubits

If the state of the system is within this space, all stabilizer measurements will give the result +1. This is known as the trivial syndrome

If the effect of errors have disturbed the logical state, taking it out of the stabilizer space, some measurements will give the result -1. This is the spoor of the errors, by which we can hope to determine how to correct them

Note: the measurements extract no information about the logical state, only the errors

Logical Pauli Operators

Consider the case of k=1, so stabilizer space is 2D

We can choose an arbitrary pair of orthogonal states within the stabilizer space to form the logical Z basis $\{|0\rangle_L, |1\rangle_L\}$

Other Pauli basis states are then defined in terms of these

$$|+\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L) \quad \text{etc}$$

Logical Pauli operators can also be defined accordingly

$$Z|0\rangle_L = |0\rangle_L, \quad Z|1\rangle_L = -|1\rangle_L, \quad X|0\rangle_L = |1\rangle_L, \quad X|1\rangle_L = |0\rangle_L, \quad \ldots$$

The Pauli bases are usually chosen such that these are also products of Paulis

They will commute with the stabilizers (since applying them remains within stabilizer space) but they'll be independent

If errors manage to implement a logical Pauli, no trace will be left in the syndrome: A logical Pauli error!

Minimum number of qubits the noise must act on to do this is called the code distance d

Basis states for stabilizer codes

A stabilizer code is a state of n qubits, and so complete basis requires 2^n states

We could choose any basis we want, like the product Z basis

$$|000\cdots\rangle, |001\cdots\rangle, |010\cdots\rangle, |011\cdots\rangle, \ldots$$

This specifies the state of the n qubits using the eigenstates of n commuting binary observables (the σ_z)

But, for stabilizer codes, there is a more natural choice of n commuting binary observables:

n-k stabilizers + k logical qubit Z's (or any other set of k commuting logical ops)

$|s_1, s_2, s_3, \ldots, s_{n-k}, z_1, z_2, \ldots, z_k\rangle$ where $s_j \in \{+1,-1\}$ is the eigenvalue of S_j

$z_j \in \{+1,-1\}$ is the eigenvalue of Z_j

$$\therefore S_j |s_1, s_2, s_3, \ldots, s_{n-k}, z_1, z_2, \ldots, z_k\rangle = s_j |s_1, s_2, s_3, \ldots, s_{n-k}, z_1, z_2, \ldots, z_k\rangle \text{, etc}$$

Stabilizer space: $\{|+1, +1, +1, \ldots, +1, z_1, z_2, \ldots, z_k\rangle\}$

Pauli Twirl Approximation

Note that the unitary operators that move between Pauli basis states are products of Paulis

$$\Pi_{i_1 \dots i_n} = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \dots \otimes \sigma_{i_n} \qquad i_1, \dots i_n \in \{0, x, y, z\}$$

$$\Pi_{i_1 \dots i_n} | S_1, S_2, S_3, \dots, S_{n-k}, z_1, z_2, \dots, z_k \rangle = | S_1', S_2', S_3', \dots, S_{n-k}', z_1', z_2', \dots, z_k' \rangle$$

$S_j = S_j'$ if $\Pi_{i_1 \dots i_n}$ commutes with S_j, $S_j = -S_j'$ otherwise

$z_j = z_j'$ if $\Pi_{i_1 \dots i_n}$ commutes with Z_j, $z_j = -z_j'$ otherwise

In general the action of a Kraus operator will not be of this form, and will create superpositions of Pauli basis states

$$|\psi\rangle = | +, +, +, \dots, +, z_1, z_2, \dots, z_k \rangle \to E_m |\psi\rangle = \sum_{S_1, \dots z_k'} C^m_{S_1, \dots, z_k'} | S_1, S_2, S_3, \dots, S_{n-k}, z_1', z_2', \dots, z_k' \rangle$$

$$= \sum_{i_1, \dots i_n} d^m_{i_1 \dots i_n} \Pi_{i_1 \dots i_n} |\psi\rangle$$

When the syndrome measurement is made, this means measurement of the $\{S_j\}$ and so measurement in this basis

The superposition created by the Kraus operator is then (mostly) destroyed

The effects of $E_m|\psi\rangle\langle\psi|E_m^\dagger = \sum_{i_1,\ldots,i_n} \sum_{j_1,\ldots,j_n} d^m_{i_1\ldots i_n} d^{m*}_{j_1\ldots j_n} \Pi_{i_1\ldots i_n} |\psi\rangle\langle\psi| \Pi_{j_1\ldots j_n}$

are then much the same as those of

$$\sum_{i_1,\ldots,i_n} |d^m_{i_1\ldots i_n}|^2 \, \Pi_{i_1\ldots i_n} |\psi\rangle\langle\psi| \Pi_{i_1\ldots i_n}$$

So any Kraus operator that is not a product of Paulis can be replaced by some that are to good approximation

$$E_m \rightarrow \{d^m_{i_1\ldots i_n} \Pi_{i_1\ldots i_n}\}$$

Combined with our previous assumption of independent environments, this means we need only worry about single qubit Kraus operators of the form

$$E_0 = \sqrt{p_0}\, \mathbb{1}, \quad E_x = \sqrt{p_x}\, \sigma_x, \quad E_y = \sqrt{p_y}\, \sigma_y, \quad E_z = \sqrt{p_z}\, \sigma_z$$

Measuring Stabilizers

Stabilizer codes have many-body operators, called stabilizers which we need to measure to detect and correct errors

But how do we measure a many body observable?

Consider a four-body observable $A = \begin{array}{ccc} & \sigma_z & \\ \sigma_z & & \sigma_z \\ & \sigma_z & \end{array} = \sigma_z^1 \sigma_z^2 \sigma_z^3 \sigma_z^4$

This has eigenvalues +/- 1

+1 eigenspace spanned by
$|0000\rangle$ $|1001\rangle$
$|0011\rangle$ $|1010\rangle$
$|0101\rangle$ $|1100\rangle$
$|0110\rangle$ $|1111\rangle$

−1 eigenspace spanned by
$|0001\rangle$ $|1000\rangle$
$|0010\rangle$ $|1011\rangle$
$|0100\rangle$ $|1101\rangle$
$|0111\rangle$ $|1110\rangle$

+1 for even number of $|1\rangle$'s, -1 for odd number

If we just wish to determine +/- 1, and don't care what happens to the state after, we can just measure in the Z basis on every qubit and see which of the above two groups the result belongs to

But this extracts more information than required, and so does not preserve superpositions

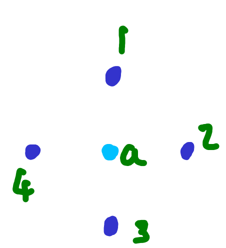
$$\left\{ \mathbb{P}_{jklm} = \frac{(\mathbb{1}+j\sigma_z^1)}{2}\frac{(\mathbb{1}+k\sigma_z^2)}{2}\frac{(\mathbb{1}+l\sigma_z^3)}{2}\frac{(\mathbb{1}+m\sigma_z^4)}{2},\ j,k,l,m \in \{+1,-1\} \right\} \neq \left\{ \mathbb{P}_{jklm} = \frac{\mathbb{1}+jklm\,A}{2} \right\}$$

e.g. $$\mathbb{P}_{++++}\left(\alpha|0000\rangle + \beta|0101\rangle + \gamma|0011\rangle\right) = \alpha|0000\rangle,$$

$$\mathbb{P}_{++++}\left(\alpha|0000\rangle + \beta|0101\rangle + \gamma|0011\rangle\right) = \left(\alpha|0000\rangle + \beta|0101\rangle + \gamma|0011\rangle\right)$$

How do we measure in a way that leaves the superpositions intact (required so that logical qubits aren't disturbed)?

We use an extra qubit to help: an ancilla



- a initially in state $|0\rangle$
- Apply $\Lambda^x_{1,a}\Lambda^x_{2,a}\Lambda^x_{3,a}\Lambda^x_{4,a}$, where $\Lambda^x_{j,a}$ with j as control, a as target
- Measure a in Z basis
- $|0\rangle$ implies $A=+1$, $|1\rangle$ implies $A=-1$

$$\Lambda^x_{c,t} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x$$

control

target

Example 1: Qubits in state $|0000\rangle$

code qubits: 1,2,3,4 — ancilla qubit: a

Start with $|0000\rangle|0\rangle$

tensor product

$\Lambda^x_{1,a}\Lambda^x_{2,a}\Lambda^x_{3,a}\Lambda^x_{4,a}$ acts trivially since all controls are $|0\rangle$

Measuring ancilla in Z basis gives $|0\rangle$, and therefore +1 with probability 1 (as required)

Example 2: Qubits in state $|1000\rangle$

Start with $|1000\rangle|0\rangle$

When $\Lambda^x_{1,a}\Lambda^x_{2,a}\Lambda^x_{3,a}\Lambda^x_{4,a}$ is applied only $\Lambda^x_{1,a}$ acts non-trivially: It applies σ_x^a

Measuring ancilla in Z basis gives $|1\rangle$, and therefore -1 with probability 1 (as required)

Same for $|0001\rangle$, $|0010\rangle$, $|0100\rangle$

Example 3: Qubits in state $|1100\rangle$

Start with $|1100\rangle|0\rangle$

When $\Lambda^{x}_{1,a}\Lambda^{x}_{2,a}\Lambda^{x}_{3,a}\Lambda^{x}_{4,a}$ is applied only $\Lambda^{x}_{1,a}$ and $\Lambda^{x}_{2,a}$ act non-trivially

Both apply σ_x^a, but since two applications of this gives the identity, net effect is trivial

Measuring ancilla in Z basis gives $|0\rangle$, and therefore +1 with probability 1 (as required)

Same for $|1010\rangle$, $|1001\rangle$, $|0110\rangle$, $|0101\rangle$, $|0011\rangle$, $|1111\rangle$

In conclusion: number of $|1\rangle$'s in state is number of times is applied to the ancilla. Even number of times results in final state $|0\rangle$, odd number results in $|1\rangle$. Measuring the ancilla then tells you whether there was an odd or even number of $|1\rangle$'s and nothing else, as required.

Does this method preserve superpositions with the eigenspaces of A?

Example 4:

$$|\psi\rangle = \underbrace{\alpha|0000\rangle + \beta|1111\rangle}_{+1 \text{ eigenspace}} + \underbrace{\gamma|0001\rangle + \delta|1000\rangle}_{-1 \text{ eigenspace}}$$

$$\Lambda^{x}_{1,a}\Lambda^{x}_{2,a}\Lambda^{x}_{3,a}\Lambda^{x}_{4,a}|\psi\rangle|0\rangle = \alpha|0000\rangle + \beta|1111\rangle)|0\rangle + (\gamma|0001\rangle + \delta|1000\rangle)|1\rangle$$

Measurement has outcome $|0\rangle$ with probability $|\alpha|^2+|\beta|^2$
State of the four qubits projected to $(\alpha|0000\rangle+\beta|1111\rangle)/(|\alpha|^2+|\beta|^2)$

Measurement has outcome $|1\rangle$ with probability $|\gamma|^2+|\delta|^2$
State of the four qubits projected to $(\gamma|0001\rangle+\delta|1000\rangle)/(|\gamma|^2+|\delta|^2)$

Superpositions are not disturbed: This is truly a measurement of A

We can similarly define B type stabilizers

$$B = \begin{array}{ccc} & \sigma_x & \\ \sigma_x & & \sigma_x \\ & \sigma_x & \end{array} = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4$$

+1 eigenspace spanned by

$|++++\rangle$ $|-++-\rangle$
$|++--\rangle$ $|-+-+\rangle$
$|+-+-\rangle$ $|--++\rangle$
$|+--+\rangle$ $|----\rangle$

-1 eigenspace spanned by

$|+++-\rangle$ $|-+++\rangle$
$|++-+\rangle$ $|-+--\rangle$
$|+-++\rangle$ $|--+-\rangle$
$|+---\rangle$ $|---+\rangle$

Now we use an ancilla in initial state $|+\rangle$, and use CNOTs with ancilla as control and others as targets

$$\Lambda^{x}_{ct} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x = \frac{1}{2}\left(\mathbb{1} + \sigma_z^c + \sigma_x^t - \sigma_z^c \sigma_x^t\right) = \mathbb{1} \otimes |+\rangle\langle +| + \sigma_z \otimes |-\rangle\langle -|$$

This shows that the interpretation of 'control' and 'target' is abitrary, because now it applies a σ_z to the 'control', based on the X basis state of the 'target'

An even or odd number of σ_z's is applied to the ancilla according to the eigenvalue of B, and the final ancilla state is $|+\rangle$ or $|-\rangle$ accordingly. So, again, measuring the ancilla gives the result for the measurement of B

Finding the states of the stabilizer space

As you all must know, σ_x^i and σ_z^i do not commute

$$\sigma_x^i\sigma_z^i = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} \qquad \sigma_z^i\sigma_x^i = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$\therefore\ \sigma_x^i\sigma_z^i = -\sigma_z^i\sigma_x^i \Rightarrow$ anti commute

But they do commute if they act on different qubits

$$\sigma_x^1 = \sigma_x \otimes \mathbb{1} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \qquad \sigma_z^2 = \mathbb{1}\otimes\sigma_z = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix} \qquad \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}$$

$\therefore\ \sigma_x^1\sigma_z^2 = \sigma_z^2\sigma_x^1 \Rightarrow$ commute

What about $\sigma_x^i\sigma_x^j$ and $\sigma_z^i\sigma_z^j$

$$\sigma_x^i\sigma_x^j\sigma_z^i\sigma_z^j = \sigma_x^i\sigma_z^i\sigma_x^j\sigma_z^j = (-\sigma_z^i\sigma_x^i)(-\sigma_z^j\sigma_x^j) = \sigma_z^i\sigma_x^i\sigma_z^j\sigma_x^j = \sigma_z^i\sigma_z^j\sigma_x^i\sigma_x^j$$

commute   anti commute   anti commute   commute

Two anticommutes make a commute!

Two commuting operators are simultaneously diagonalizable

$$[M,N]=0 \Rightarrow M=\sum m|m\rangle\langle m|, \quad N=\sum n|m\rangle\langle m|$$

same

So this is true for $\sigma_x^i\sigma_x^j$ and $\sigma_z^i\sigma_z^j$

A state can be in: +1 eigenspace of both
+1 eigenspace of one, -1 of the other
-1 eigenspae of both

But since the single qubit operators don't commute, such states cannot be product states. They must be entangled

+1 eigenspace of $\sigma_x^i\sigma_x^j$ spanned by $|{+}{+}\rangle, |{-}{-}\rangle$

+1 eigenspace of $\sigma_z^i\sigma_z^j$ spanned by $|00\rangle, |11\rangle$

+1 eigenspace of both $\sigma_x^i\sigma_x^j$ and $\sigma_z^i\sigma_z^j$

$$\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)=\frac{1}{\sqrt{2}}\left(|{+}{+}\rangle+|{-}{-}\rangle\right)=|\phi^+\rangle$$

$|\psi^-\rangle$ is in -1 eigenspace of both

$|\phi^-\rangle$ is in +1 eigenspace of $\sigma_z^i\sigma_z^j$, -1 of $\sigma_x^i\sigma_x^j$

$|\psi^+\rangle$ is in -1 eigenspace of $\sigma_z^i\sigma_z^j$, +1 of $\sigma_x^i\sigma_x^j$

To find such mutual eigenstates, we first find an eigenstate of one

$|00\rangle$ is a $+1$ eigenstate of $\sigma_z^i \sigma_z^j$

And a projector to an eigenspace of another

$\mathbb{P}_x^+ = \frac{\mathbb{1} + \sigma_x^i \sigma_x^j}{2} = |{+}{+}\rangle\langle{+}{+}| + |{-}{-}\rangle\langle{-}{-}|$ projects onto $+1$ eigenspace of $\sigma_x^i \sigma_x^j$

Then apply the latter to the former $|\lambda^{++}\rangle = \mathbb{P}_x^+ |00\rangle$

The result then belongs to the required eigenspaces

commute

$$\sigma_z^i \sigma_z^j |\lambda^{++}\rangle = \sigma_z^i \sigma_z^j \mathbb{P}_x^+ |00\rangle = \sigma_z^i \sigma_z^j \frac{1}{2}\left(\mathbb{1} + \sigma_x^i \sigma_x^j\right)|00\rangle = \frac{1}{2}\left(\mathbb{1} + \sigma_x^i \sigma_x^j\right)\sigma_z^i \sigma_z^j |00\rangle$$

commute

$$= \frac{1}{2}\left(\mathbb{1} + \sigma_x^i \sigma_x^j\right) \times (+1) \times |00\rangle = (+1) \times \mathbb{P}_x^+ |00\rangle = (+1)|\lambda^{++}\rangle$$

$$\sigma_x^i \sigma_x^j |\lambda^{++}\rangle = \sigma_x^i \sigma_x^j \mathbb{P}_x^+ |00\rangle = \sigma_x^i \sigma_x^j \frac{1}{2}\left(\mathbb{1} + \sigma_x^i \sigma_x^j\right)|00\rangle = \frac{1}{2}\left(\sigma_x^i \sigma_x^j + \left(\sigma_x^i \sigma_x^j\right)^2\right)|00\rangle$$

$$= \frac{1}{2}\left(\sigma_x^i \sigma_x^j + \mathbb{1}\right)|00\rangle = (+1) \times \mathbb{P}_x^+ |00\rangle = (+1)|\lambda^{++}\rangle$$

Note that there are four ways we could've tried to get $|\lambda^{++}\rangle$

$$P_x^+|00\rangle = \frac{1}{2}\left(\mathbb{1} + \sigma_x^i\sigma_x^j\right)|00\rangle = \frac{1}{2}\left(|00\rangle + |11\rangle\right)$$

$$P_x^+|11\rangle = \frac{1}{2}\left(\mathbb{1} + \sigma_x^i\sigma_x^j\right)|11\rangle = \frac{1}{2}\left(|11\rangle + |00\rangle\right) = \frac{1}{2}\left(|00\rangle + |11\rangle\right)$$

$$P_z^+|++\rangle = \frac{1}{2}\left(|++\rangle + |--\rangle\right) = \frac{1}{4}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle\right) = \frac{1}{2}\left(|00\rangle + |11\rangle\right)$$

$$P_z^+|--\rangle = \frac{1}{2}\left(|--\rangle + |++\rangle\right) = \frac{1}{2}\left(|00\rangle + |11\rangle\right)$$

In all cases we get the same result

This is not true in general, but only when subspaces are 1D