Quantum Noise

(N+C Chapter 8)

If you have a qubit in a given state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = a|+\rangle + b|-\rangle$ put it in a box and send it to Australia, will it still be in the state $|\psi\rangle$ when it gets there?

We would expect that, over such a long time, it will have decohered to become a mixed state such as

$$|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|, \quad |a|^2|+\rangle\langle +| + |b|^2|-\rangle\langle -|, \quad \tfrac{1}{2}\mathbb{1}$$

We would expect a similar thing for Schroedinger's cat

$$|+\rangle \otimes |alive\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|alive\rangle + |1\rangle\otimes|alive\rangle\right)$$

$$\xrightarrow{U} \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|alive\rangle + |1\rangle\otimes|dead\rangle\right)$$

$$\xrightarrow{\text{decoherence}} \frac{1}{2}\left(|0,alive\rangle\langle 0,alive| + |1,dead\rangle\langle 1,dead|\right)$$

Even within a box, interactions between the cat and the environment would destroy the superposition

But what exactly is the noise that causes this, and how can it be described?

What sort of process in QM can take a single qubit pure state and make a single qubit mixed state?

Within the postulates of QM, we can find two things that change a state: measurement and unitary evolution

If someone borrows our qubit in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, measures it in the Z basis and then gives it back without telling us the result, we then have a qubit described by

$$|\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$$

So decoherence could be caused by Gremlins who keep measuring our qubits and keeping the results secret

In a way it is. But in another more accurate way, it isn't.

What about unitary evolution?

Suppose our qubit is a spin-1/2 and feels a magnetic field

$$H = B\sigma_z$$

It will then precess

$$|\psi(t)\rangle = e^{-iB\sigma_z t}|\psi\rangle = e^{-iBt}\alpha|0\rangle + e^{iBt}\beta|1\rangle$$

$$|\psi(t)\rangle\langle\psi(t)| = \begin{pmatrix} |\alpha|^2 & e^{-2iBt}\alpha\beta^* \\ e^{2iBt}\alpha^*\beta & |\beta|^2 \end{pmatrix}$$

If we don't know how long it has been precessing, we then have some uncertainty about the state

$$\rho = \int_{t=0}^{\pi/B} |\psi(t)\rangle\langle\psi(t)|\, dt = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\int e^{-2iBt}dt \\ \alpha^*\beta\int e^{2iBt}dt & |\beta|^2 \end{pmatrix} = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

Using $\int_0^{2\pi} e^{i\theta}\, d\theta = 0$

The qubit might also interact with another, resulting in a unitary evolution that entangles them

Initial state: $|\psi\rangle \otimes |0\rangle$

our qubit (s) environment qubit (e)

$$H_{s,e} = \frac{1}{2}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_x - \sigma_z\otimes\sigma_x\right),$$

$$U(t) = e^{-iH_{s,e}t}, \qquad U\left(\frac{\pi}{2}\right) = -i\,\Lambda^x, \qquad \Lambda^x = |0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\sigma_x$$

Again, this results in a mixed state

$$|\psi(\tfrac{\pi}{2})\rangle = U\left(\frac{\pi}{2}\right)|\psi\rangle\otimes|0\rangle = -i\,\Lambda^x\left(\alpha|00\rangle + \beta|10\rangle\right) = -i\left(\alpha|00\rangle + \beta|11\rangle\right)$$

$$\rho_s = tr_e\left(|\psi(\tfrac{\pi}{2})\rangle\langle\psi(\tfrac{\pi}{2})|\right) = \alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1|$$

For a different Hamiltonian, we would get different decoherence

$$H_{s,e} = \mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_x - \sigma_x\otimes\sigma_x \implies \rho_s = a|+\rangle\langle +| + b|-\rangle\langle -|$$

So we have three mechanisms that cause decoherence

Measurement by Gremlins

Hamiltonian acting on the qubit in unknown way

Hamiltonian interacting qubit with enviroment

The first and third can be combined: Gremlin's measurement device is part of environment, and they use an interaction Hamiltionian to make the measurement

All three can then be combined into one: The thing that disturbs our qubits is the action of an unwanted Hamiltonian

$$H = H_s + H_e + H_{s,e}\,, \qquad \rho_s(t) = tr_e\left(e^{-iHt}|\psi\rangle\otimes|e_0\rangle\langle\psi|\otimes\langle e_0|e^{iHt}\right)$$

This description requires us to consider the enviroment, its initial state $|e_0\rangle$, its internal interactions H_e, etc.

A very complex description. Can we simplify it to find only the effects on S?

Operator Sum Representation

Consider the more general case that our system, S, and the environment, E, are more than just qubits

Our system is initially in a state ρ_S. In general this could be mixed, as it might be part of states entangled to our friends

The enviroment is in an initial state $|e_0\rangle$, which can be pure in general because we can just add extra enviroment to purify it if not. We define a orthonormal basis $\{|e_m\rangle\}$ that includes the initial state

System and environment interact according to a Hamiltonian $H_{SE}(t)$ which induces a unitary evolution

$$U(t) = e^{-i\int H_{SE}(t)\,dt} = \sum_{ijkl} u_{ijkl} |s_i e_k\rangle\langle s_j e_l|$$

State of S at time t is then

$$\rho_S(t) = \mathrm{tr}_E\left(U(t)\left[\rho_S \otimes |e_0\rangle\langle e_0|\right]U^\dagger(t)\right)$$

$$= \sum_m \langle e_m|\left(U(t)\,|s_i e_k\rangle\langle s_j e_l|\right)\left[\rho \otimes |e_0\rangle\langle e_0|\right]\left(U^\dagger(t)\,|s_i e_k\rangle\langle s_j e_l|\right)|e_m\rangle$$

$$= \sum_m E_m \rho E_m^\dagger \qquad E_m \equiv \langle e_m|U|e_0\rangle \text{ is an operator on } \mathcal{H}_S$$

$$\mathrm{tr}\left[\rho_S(t)\right] = 1 \quad \Rightarrow \quad \sum_m E_m E_m^\dagger = \mathbb{1}_S$$

This describes any kind of time evolution that can happen to S (as long as we extract no information)

The effects of noise can then be described by a superoperator, an operator that acts on density operators, of the form

$$\mathcal{E}(\rho) = \sum_m E_m \rho E_m^\dagger \qquad \sum_m E_m^\dagger E_m = \mathbb{1}_S$$

The E_m are known as the operation elements or Kraus operators

This may also be expressed

$$\mathcal{E}(\rho) = \sum_m p_m \rho_m \qquad p_m = tr(E_m \rho E_m^\dagger) \qquad \rho_m = E_m \rho E_m^\dagger / p_m$$

So we can think of noise as a classical process that replaces ρ with ρ_m with probability p_m

The representation of a noise process in terms of Kraus operators is not necessarily unique

In general, Kraus operators are not unitary, hermitian, projectors or any other nice thing. Their time-dependence can also be complicated

One simplification we usually assume for many-body systems is that each particle interacts with a different environment. So the effect of noise on each particle is uncorrelated

$$\mathcal{E}(\rho) = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_n} E_{m_1} \otimes E_{m_2} \otimes \cdots \otimes E_{m_n} \, \rho \, E_{m_1}^\dagger \otimes E_{m_2}^\dagger \otimes \cdots \otimes E_{m_n}^\dagger$$