Entang entropy

✓ → Schmidt (see how Schmidt get S)

Renyi

✓ → Entanglement entropy (estimate S_2 using random mes)

Other measure

→ concurence _ negativity

PPT crit

→ separability

Sharing of entanglement

→ Monogamy

Propagat° of entanglement

→ Area law

Testing existance of Q:

→ Bell tests.

Entanglement witness

→ Ent_ witness

LOCC cannot create entanglement

→ locc.

Remarkable entangled state

→ GHZ_ W

Different pattern in entanglement:

→ entanglement structure

→ use : → stabilizer

Regenerating entanglement:

→ purification.

Teleportat° w/ noise.

→ telle

Choi channel

→ Choi

Noise model

→ Noise _ entanglement

Competitor of QC

→ tensor _ network

Notion of entanglement theory

0. Reminder

ρ is density matrix of state

- equivalent repres. of isolated sys. w/ state vector
- said to be the only valid one for open: system

. Properties:

- $\rho^{\dagger} = \rho$
- $\mathrm{Tr}(\rho) = 1$
- $\mathrm{Tr}(\rho^2) \leq 1$; $= 1$ iff pure state

  iff $\exists |\psi\rangle$ s.t. $\rho = |\psi\rangle\langle\psi|$
- definite positive
- diag are probabilities

. Remarkable ρ:

- pure
- mixed
- thermal mix
- classical mix

I Evaluating entanglement

1| Motivation : notion of locality

Superposit° vs entanglement , a notion of locality

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(\underbrace{|00\rangle}_{\equiv|0\rangle} + \underbrace{|11\rangle}_{\equiv|3\rangle}\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |3\rangle\right) \Rightarrow \text{only superposition?}$$

$\Rightarrow$ no because of <u>locality / separability</u>:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

spatially wh 1 locat° (superposition within)

spatiatially in another

entanglement without

We say " Qubit 1 entangled w/ qubit 2 "

How to evaluate : Von Neumann entanglement entropy.

2] Partial trace: an example

ex: $\rho_{\phi^+} = |\phi^+\rangle\langle\phi^+|$

$$= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$= \frac{1}{2}\begin{array}{cc}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}\end{array}$$

$\mathrm{Tr}_2 \downarrow$

Basis of $\mathcal{H}_2$

$$\langle 0_2|\rho_{\phi^+}|0_2\rangle + \langle 1_2|\rho_{\phi^+}|1_2\rangle$$

$$\begin{aligned} = \frac{1}{2}\Big(& \mathbb{1}\otimes\langle 0|\,|00\rangle\langle 00|\,\mathbb{1}\otimes|0\rangle \\ & + \mathbb{1}\otimes\langle 0|\,|00\rangle\langle 11|\,\mathbb{1}\otimes|0\rangle \\ & + \mathbb{1}\otimes\langle 0|\,|11\rangle\langle 00|\,\mathbb{1}\otimes|0\rangle \\ & + \mathbb{1}\otimes\langle 0|\,|11\rangle\langle 11|\,\mathbb{1}\otimes|0\rangle \\ & + \mathbb{1}\otimes\langle 1|\,|00\rangle\langle 00|\,\mathbb{1}\otimes|1\rangle \\ & + \mathbb{1}\otimes\langle 1|\,|00\rangle\langle 11|\,\mathbb{1}\otimes|1\rangle \\ & + \mathbb{1}\otimes\langle 1|\,|11\rangle\langle 00|\,\mathbb{1}\otimes|1\rangle \\ & + \mathbb{1}\otimes\langle 1|\,|11\rangle\langle 11|\,\mathbb{1}\otimes|1\rangle \Big) \end{aligned}$$

$$\begin{aligned} = \frac{1}{2}\Big(& |0\rangle\langle 0| \\ & + 0 \\ & + 0 \\ & + 0 \\ & + 0 \\ & + 0 \\ & + 0 \\ & + |1\rangle\langle 1| \Big) \end{aligned}$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \rho_1$$

partial trace

reduced density mat

3) Partial trace : general

* In general : $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$\begin{array}{cc} \uparrow & \uparrow \\ |A_i\rangle & |B_j\rangle \end{array}$$

$$\rho_A = \mathrm{Tr}_B(\rho) = \sum_j \mathbb{1} \otimes \langle B_j| \rho (\mathbb{1} \otimes |B_j\rangle)$$

$$\rho_B = \mathrm{Tr}_A(\rho) = \sum_i \langle A_i| \otimes \mathbb{1} \; \rho \; |A_i\rangle \otimes \mathbb{1}$$

... we can partial trace more.
rq: $\rho_A = \rho_A^\dagger$; $\mathrm{Tr}\, \rho_A = 1$; etc...

* Remarks:

- If initially pure, $\rho_{reduced}$ can be mixed etc.
- if in " , is mixed A & B are entangled.
- if initially pure, $\rho_{reduced}$ pure → A & B not entangled → the state is separable and

$$\rho = \rho_A \otimes \rho_B \qquad \text{(prove it)}$$

- if initially mixed, ρ_A : pure → A separable & only B is mixed (not interesting)
- if mixed mixed → A ? B

→ need a measure → VN ent. ent.

4] Bipartite Von Neumann entanglement entropy

A] Formula

Von Neumann entang. entropy (bipartite)

$$S_{A/B} = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right)$$
$$= -\mathrm{Tr}_B\left(\rho_B \log \rho_B\right)$$

Meaning of $\log \rho_A$: $\rho_A = \rho_A^{+} \Rightarrow$ diago

i.e. $\log(\rho_A) = \log(U D U^{+})$

$$= U \log(D)\, U^{+}$$

$$= U \log\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^{+}$$

$$= U \begin{pmatrix} \log\lambda_1 & & \\ & \ddots & \\ & & \log\lambda_n \end{pmatrix} U^{+}$$

What if $\lambda_i = 0$?

$$\Rightarrow \mathrm{Tr}\left(\rho_A \log \rho_A\right) = \mathrm{Tr}\Big(U \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} \underbrace{U^\dagger U}_{1}$$

abuse of notat° for explanation

$$\hookrightarrow \cdot \begin{pmatrix}\log\lambda_1 & & \\ & \ddots & \\ & & \log\lambda_n\end{pmatrix} U^\dagger\Big)$$

$$= \mathrm{Tr}\left(\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\begin{pmatrix}\log\lambda_1 & & \\ & \ddots & \\ & & \log\lambda_n\end{pmatrix} \cdot U^\dagger U\right)$$

cyclic rotation in the trace

$$= \mathrm{Tr}\begin{pmatrix}\lambda_1\log\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\log\lambda_n\end{pmatrix}$$

$$= \sum_i \lambda_i \log \lambda_i$$

if 0, cancel the infinite

$$= \sum_{i \mid \lambda_i \neq 0} \lambda_i \log \lambda_i$$

$1 \geq \lambda_i \geq 0$ by def.

B] A measure of entanglement

$S_{A/B}$ <u>is</u> a measure of entanglement i.e.

× non-negativity : $S_{A/B} \geq 0$

× $S_{A/B} = 0$ iff $\rho = \rho_A \otimes \rho_B$ (ρ_A, ρ_B pure)

× Monotonically decrease under LOCC (local operation and classical communication)

↳ i.e. operation that, well, do not increase entanglement.

i.e. operation that you can do on A or B / the operation where you need both A & B to do.

i.e. separable operation

i.e. operation to cannot turn a separable state into a non separable one.

→ see «2_locc.ipynb», but not the 1st one to check

C] Examples

ex $$S_1(\phi^+) = -\text{Tr}\left(\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\log\left(\frac{1}{2}\begin{pmatrix}1&\\&1\end{pmatrix}\right)\right)$$

$$= \text{Tr}\begin{pmatrix}\frac{1}{2}\log 2 & \\ & \frac{1}{2}\log 2\end{pmatrix}$$

$$= \log 2$$: the max that can be obtained w/ 2 qubit

$\Rightarrow$ Bell pair are "maximally entangled states"

$S \to$ not measurable

D] Schmidt decomposition & SVD

* Schmidt decomp & max entang. (cf Schmidt-decomp)

  If $\dim \mathcal{H}_A = N_A$; $\dim \mathcal{H}_B = N_B$, then

  if $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \Rightarrow \exists \; \alpha_i \;, \; i \leq \min\{N_1, N_2\}$

  s.t. $|\psi\rangle = \sum_i \alpha_i \; |\phi_i\rangle \otimes |\varphi_i\rangle$ where

  $\{|\phi_i\rangle\}_{i \leq N_A}$ basis of $\mathcal{H}_A$; $\{|\varphi_i\rangle\}_{i \leq N_B}$ basis of $\mathcal{H}_B$

  $\Rightarrow S_{A/B} \leq \log(\min\{N_A, N_B\})$

  i.e. a qubit entangled w/ the universe ... only shares a max of $\log 2$ entanglement w/ it.

* there are several algo to get the α_i, but the Singular value decomposition is a classic:

  i.e. if M matrix (not necessarly square):

  $\exists \; U, V$ unitary s.t.

  $$M = U \, D \, V^{-1}$$

  D diagonal not square.

  why ? $|\phi_i\rangle \otimes |\varphi_i\rangle \rightarrow |\phi_i\rangle\langle\varphi_i|$ in the Schmidt decomposition.

  → see <<1 Schmidt decomposition>>

5] Renyi-entropies

Renyi entropy : $S_\alpha(\rho) = \frac{1}{1-\alpha} \log(\rho_A^\alpha) \quad \alpha \geq 0$

estimatable

$\alpha \to 0 \to$ capture all non-zero terms & put it to 1.

$$\rho_A^\alpha = e^{\alpha \log \rho_A} = U e^{\alpha \log(D)} U^{-1}$$

$$= U \prod_{\substack{i \\ \lambda_i \neq 0}} e^{\alpha \log \lambda_i} U^{-1}$$

$$= U \begin{pmatrix} \lambda_i^\alpha & \\ & 0 \end{pmatrix} U^{-1}$$

$$\underset{\alpha \to 0}{=} U \frac{1}{N} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} U^{-1}$$

quick & dirty

$\alpha \to \infty$: "rich get richer"

$\alpha \to 1$: coincide w/ VNEE.

$\to$ see << 3_ Renyi-entropy >> for estimation

6) Other measures

A) Concurrence (bipartite)

$$C_A(\rho) = \sqrt{2(1 - \mathrm{Tr}(\rho_A^2))}$$

For 2 qubits: $C(\rho) \equiv \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$

where $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ v.p. of $R = \sqrt{\sqrt{\rho}\,\tilde{\rho}\sqrt{\rho}}$; $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\,\rho^*(\sigma_y \otimes \sigma_y)$

in Z basis

B) Negativity

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_A}\|_1 - 1}{2}$$

$$= \sum_i \frac{|\lambda_i| - \lambda_i}{2}$$

Not a density mat; can have $\lambda < 0$

ρ^{T_A} : partial transpose

$\|X\|_1 = \mathrm{Tr}(|X|) = \mathrm{Tr}\sqrt{X^\dagger X}$

→ see «4_Concurrence_negativity»

C) Log negativity

$$E_N(\rho) = \log_2 \|\rho^{T_A}\|_1 = \log_2(2\mathcal{N} + 1)$$

D) Mutual information (tripartite)

* Not an entanglement monotone

$$I(A:B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB})$$

$$= S(\rho^{AB} \| \rho^A \otimes \rho^B)$$

III Estimating & evaluating entanglement

1] Confrontation with hidden variables

ex : teleportation ; is there really no Δ^{able} that we have no access to that says in advance that Alice's envelope was not meant to be red all along, & Bob's envelope blue.

→ can build a mathematical theory out of it

→ Has ≠ predict° than QM

→ QM was tested right (Nobel)

2] CHSH & Bell inequalities (a stat. constraint)

Clauser - Horne - Shimony - Holt

$|S| \leq 2$ if hidden Δ^{able} ($\leq 2\sqrt{2}$ if Q.)

w $S = E(a,b) - E(a,b') + E(a',b) + E(a',b')$

In QM of course

$$\begin{cases} E(a,b) = \langle \psi(a,b) | Z \otimes Z | \psi(a,b) \rangle \\ |\psi(a,b)\rangle = \underbrace{A(a) \otimes B(b)}_{\text{define a precise state of the device}} |\psi_{ref}\rangle \end{cases}$$

$\hookrightarrow = \text{e.g. } |00\rangle$

$$\rightarrow E(a,b) = \frac{N_{++} - N_{+-} - N_{-+} + N_{--}}{N_{++} + N_{+-} + N_{-+} + N_{--}}$$

↳ a, b are the "hidden Δ^{able}"

See << 5_ Bell_test >>

3) Entanglement witnesses

An entanglement witness W is an operator / Fct^{al} that is $\oplus$ on all separable (biseparable) states.

If strictly negative, it is an entangled state.

↳ not iff.

→ see << 6_entanglement_witness >>

4) PPT criteria (an entang. witness)

ex PPT criteria

· requirements: partial trace:

$$\rho = \Sigma\, \alpha_{ijkl}\, |i\rangle\langle j| \otimes |k\rangle\langle l|$$

T_B : partial trace

$$\rho^{T_B} = \Sigma\, \alpha_{ijkl}\, |i\rangle\langle j| \otimes |l\rangle\langle k|$$

$$= \Sigma\, \underline{\underline{\alpha_{ijlk}}}\, |i\rangle\langle j| \otimes |k\rangle\langle l|$$

PPT : if ρ separable, v.p. of $\rho^{T_B} \in \mathbb{R}^+$

hence if $\exists \lambda_i < 0$ v.p. of ρ^{T_B}, then ρ entangle

note: v.p. of $\rho^{T_B} \in \mathbb{R}^+ \not\Rightarrow \rho$ separable

ρ entangles $\not\Rightarrow \exists \lambda_i < 0$ v.p. of ρ^{T_B}

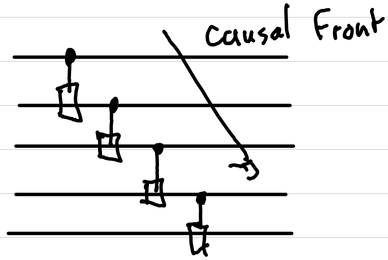
→ See << 7_separability >>

IV Entanglement pattern

1) Propagation of entanglement

Typically, through 2-by-2 local interaction.

2 patterns:



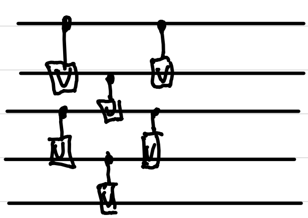
After a single front

1 entangled w/ N, but

1 is not modified by N

past state (while N is)

→ need a return



Local constant interact:

Front the second layer onwards,

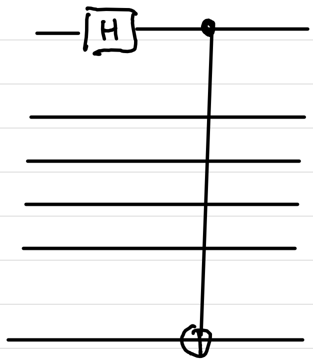
Vanishing entanglement btw

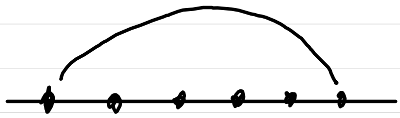
1 & N. But no causal link.

→ see « 8_Area_law »

2] Range of entanglements

A]





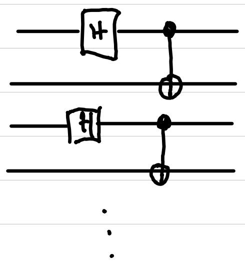
→ long range

any S convex locally $\neq 0$

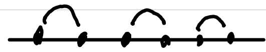
but actually not a lot of entg.

(separable, albeit not completely)

B]



} 2-by-2 entang.



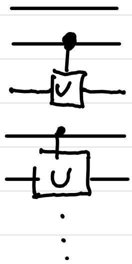
Some $S=0$, some only as much as above.

→ short-range

(separable, albeit not completely)

C]

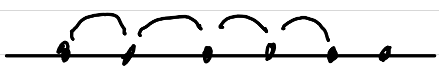
Same



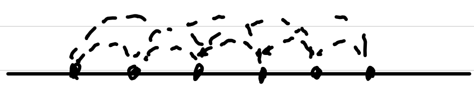
→ not separable, $S \neq 0$ ever.

but not all entangled state are possible

e.g. the 1st qubit is vanishingly entangled w/ the last



D] Possible to entangle at longer range, but at the cost of shorter range:



→ see 9_entanglement_structure

10_GHZ_W

3] Entanglement distillation & purification

A & B bought $|\phi^+\rangle$ at Carlie, not Charlie!

They have (copies of) $\rho = F|\phi^+\rangle\langle\phi^+| + \frac{1-F}{3}\left(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|\right)$

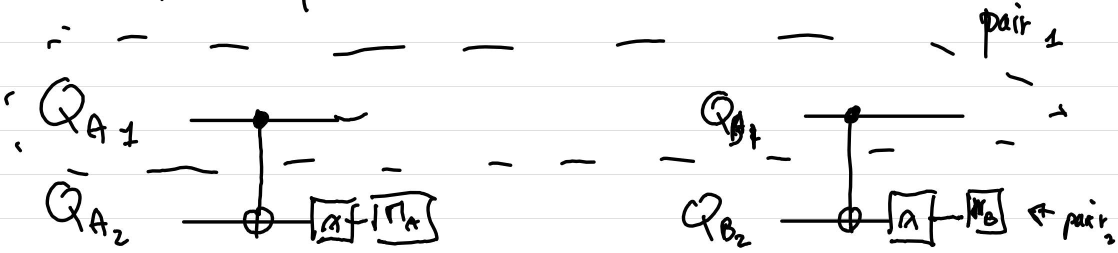
w/ $F > 0{,}5$

How can they do better, knowing that they cannot meet? (Ideally, they want $|\phi^+\rangle\langle\phi^+|$ only)

→ BBPSSW protocols.

Idea: sacrifying copies (a. distillation) to increase the purity of the other (purification)

Ex w/ 2 copie



Alice calls Bob. If $M_A = M_B$ → they keep pair 1. Otherwise, they throw it.

The pairs left look like:

$$\rho' = F'|\phi^+\rangle\langle\phi^+| + \frac{1-F'}{3}(\dots)$$

w/ $1 > F' > F$.

→ see «12_ Purification»

V Conclusions

- Entanglement distinguishes itself from superposition in terms of locality
- When two part of a system are not entangled, we say that they are separable / biseparable
- Entanglement, however, is an abstract qty difficult to capture. It can be theoretically measured using an entanglement measure ... but only few of them can be estimated w/ actual system measurements.
- More often, we use entanglement witnesses or entang. criteria, that are + accessible, but less informative.
- Using measure or witnesses, it is possible to have some understanding of patterns of entanglement existing in a state. In QC, these patterns are directly related to the shape of the circuit when the depth is small!
- Entanglement leads to non-classical correlations. Not causality!
- « Recuperating » info from entanglement can be called entanglement distillat°. It is costly in terms of states.
- Entanglement is exclusively initiated locally
- ... but can be enhanced at a distance w/ purificat°.