Hint 1

For a repetition code which stores a $|+\rangle$ as $|+\rangle \otimes |+\rangle \otimes |+\rangle$ and a $|-\rangle$ as $|-\rangle \otimes |-\rangle \otimes |-\rangle$. The stabilizers are

$$S_1 = \sigma_x^1 \sigma_x^2 \quad , \quad S_2 = \sigma_x^2 \sigma_x^3$$

Logical Z must have the effect

$$Z\,|+\rangle_3 = |-\rangle_3 \quad , \quad Z\,|-\rangle_3 = |+\rangle_3$$

$$\therefore Z = \sigma_z^1 \sigma_z^2 \sigma_z^3$$

Logical X must have the effect

$$X|+\rangle_3 = |+\rangle_3 \quad , \quad X|-\rangle_3 = -|-\rangle_3$$

$$\therefore \quad X = \sigma_x^1 \text{ (or } \sigma_x^2 \text{ or } \sigma_x^3\text{)}$$

For one storing $|0\rangle$ as $|000\rangle$ and $|1\rangle$ as $|111\rangle$

$$S_1 = \sigma_z^1 \sigma_z^2 \quad S_2 = \sigma_z^2 \sigma_z^3$$

$$Z = \sigma_z^1 \quad , \quad X = \sigma_x^1 \sigma_x^2 \sigma_x^3$$

(end of hint)

Hint 2:

For the repetition code: $|+\rangle \rightarrow |+++\rangle, \ |-\rangle \rightarrow |---\rangle$

Since

$$X = \sigma_x^1 \text{ or } \sigma_x^2 \text{ or } \sigma_x^3$$

A single σ_x error is enough to cause a single logical X error. It will be undetected by the stabilizers

However a single σ_z will have the effect

$$\sigma_z^1|+\rangle = |-++\rangle \quad \sigma_z^1|-\rangle = |+--\rangle$$

The check op S1 will see that the first and second qubit states aren't the same, but the second and third are. The possibilities are that a z occurred on onlt the first qubit, or that it occurred on the second and third. The latter is clearly less likely, so a z on the first is assumed and corrected accordingly

$$\sigma_z^1|-++\rangle = |+\rangle \quad \sigma_z^1|+--\rangle = |-\rangle$$

The same for a single z on the second or third qubit, so single z errors can be detected and corrected

For two z, however, correction cannot be achieved. For example, consider a z on 2 and 3

$$\sigma_z^2 \sigma_z^3 |-\rangle = |-++\rangle \qquad \sigma_z^2 \sigma_z^3 |+\rangle = |+--\rangle$$

S1 and S2 see the same thing as the single z on the first qubit. As in that case, we conclude that a single z on the first is most likely, and correct accordingly by applying what we think is another z on the first to cancel it out

$$\sigma_z^1 \left(\sigma_z^2 \sigma_z^3 |-\rangle \right) = Z|-\rangle = |+\rangle$$

This means that we end up applying a logical Z! We consolidate the logical error rather than correcting it!

For a z error on each qubit the noise has applied a logical error, and so cannot be corrected

For the repetition code with

$$|0\rangle \to |000\rangle \qquad |1\rangle \to |111\rangle$$

everything is the same except that the roles of x and z are interchanged

a) Consider 9 qubits. We store one logical qubit in the first 3, another in the second 3 and a third in the third 3. In all cases the encoding

$$|+\rangle_3 = |+++\rangle, \quad |-\rangle_3 = |---\rangle$$

is used. The stabilizers and logical operators are then

$$S_1 = \sigma_x^1 \sigma_x^2, \quad S_2 = \sigma_x^2 \sigma_x^3, \quad X_1 = \sigma_x^1, \quad Z_1 = \sigma_z^1 \sigma_z^2 \sigma_z^3$$

$$S_3 = \sigma_x^4 \sigma_x^5, \quad S_4 = \sigma_x^5 \sigma_x^6, \quad X_2 = \sigma_x^4, \quad Z_7 = \sigma_z^4 \sigma_z^5 \sigma_z^6$$

$$S_5 = \sigma_x^7 \sigma_x^8, \quad S_6 = \sigma_x^8 \sigma_x^9, \quad X_3 = \sigma_x^7, \quad Z_3 = \sigma_z^7 \sigma_z^8 \sigma_z^9$$

Now we use these 3 logical qubits to store one, using the encoding

$$|0\rangle_9 = |0\rangle_3 |0\rangle_3 |0\rangle_3, \quad |1\rangle_9 = |1\rangle_3 |1\rangle_3 |1\rangle_3$$

This gives us additional stabilizers made out of the logical operators above

$$S_7 = Z_1 Z_2 = \sigma_z^1 \sigma_z^2 \sigma_z^3 \sigma_z^4 \sigma_z^5 \sigma_z^6$$

$$S_8 = Z_2 Z_3 = \sigma_z^4 \sigma_z^5 \sigma_z^6 \sigma_z^7 \sigma_z^8 \sigma_z^9$$

b) The logical operators are now

$$X = X_1 X_2 X_3 = \overset{\text{or 2, or 3}}{\overset{\downarrow}{\sigma_x^{1}}}\,\overset{\text{or 5, or 6}}{\overset{\downarrow}{\sigma_x^{4}}}\,\overset{\text{or 8, or 9}}{\overset{\downarrow}{\sigma_x^{7}}}$$

$$Z = Z_1 = \sigma_z^{1}\sigma_z^{2}\sigma_z^{3} \quad (\text{or } Z_2, \text{ or } Z_3)$$

All are 3-body operators. So code is distance 3

c) σ_x occurs with prob p_x and σ_z with p_z

For a code

$$|+\rangle \to |+++\rangle, \quad |-\rangle \to |---\rangle$$

- Logical X error after correction requires an odd number of σ_x errors

- Logical Z error after correction requires 2 or 3 σ_z errors

So error rates after the first level are

$$p_x' = 3p_x(1-p_x)^2 + p_x^3 \simeq 3p_x$$

$$p_z' = 3p_z^2(1-p_z) + p_z^3 \simeq 3p_z^2$$

For the next level

$$|+\rangle \to |+++\rangle, \quad |-\rangle \to |---\rangle$$

So the roles of X and Z are changed

$$P_X = 3 P_x'^2 (1-P_x) + P_x^3 \approx 3 P_x'^2 \approx 27 P_x^2$$

$$P_Z = 3 P_Z' (1-P_Z)^2 + P_Z^3 \approx 3 P_Z' \approx 9 P_Z^2$$