1

a) 9 level (l-1) per level (l)

$\therefore$ $n(l) = 9^l$

b) level 1 is distance 3
=> operations required on 3 level 0 qubits

level 2 is distance 3
=> operations required on 3 level 1 qubits
=> operations required on 3^2 level 0 qubits

So for level l we need to act on 3^l level 0 qubits for a logical operation

$d = 3^l$

c) $$P_x^{(l)} < 27\left(P_x^{(l-1)}\right)^2 = \frac{\left(27 P_x^{(l-1)}\right)^2}{27}$$

$$P_x^{(1)} < 27 P_x^2$$

$$P_x^{(2)} < \frac{\left(27 \times 27 P_x^2\right)^2}{27} = \frac{\left(27 P_x\right)^4}{27}$$

$$P_x^{(3)} < \frac{\left(27 \times \frac{1}{27}\left(27 P_x\right)^4\right)^2}{27} = \frac{\left(27 P_x\right)^8}{27}$$

$$P_x^{(l)} < \frac{\left(27 P_x\right)^{2^l}}{27}$$

$$n(l) = 9^l = 2^{l \log 9} \quad \therefore 2^l = n(l)^{1/\log 9}$$

$$\therefore \quad P_x^{(l)} < \frac{\left(27 P_x\right)^{n(l)^{1/\log 9}}}{27}$$

exponential decay with $n(l)^\beta$ for

$P_x < 1/27$

$\beta = 1/\log 9$

d) Same as (c), but using

$$p_z^{(l)} < 9\left(p_z^{(l-1)}\right)^2$$

2 The code has n physical qubits and k logical qubits

The probability distribution for the errors on each physical qubit is

$p_0, p_x, p_y, p_z \quad (p_0 = 1 - p_x - p_y - p_z)$

The number of bits required to store the information about which errors occurred is then

$$n\,H[p_0, p_x, p_y, p_z] = -n\left(p_0 \log p_0 + p_x \log p_x + p_y \log p_y + p_z \log p_z\right)$$

With logs taken base 2

n qubits can store n bits, so this information can be stored in the n qubits (along with the k logical qubits) if

$$n \geqslant k + n\, H[p_0, p_x, p_y, p_z]$$

$$\therefore\ H[p_0, p_x, p_y, p_z] \leqslant \frac{n-k}{n} \to 1$$

b) In this case

$$H[p_0, p_x, p_y, p_z] = 2H(p)$$

$\therefore$ the bound is

$$H(p) \leqslant \frac{1}{2} \implies p_c \simeq 11\%$$

(they can get Wolfram alpha to do this bit)

c) $$H[p_0, p_x, p_y, p_z] = -\left((1-p)\log(1-p) + 3\left[\frac{p}{3}\log\frac{p}{3}\right]\right)$$

$$= -\left((1-p)\log(1-p) + p\log p - p\log 3\right)$$

$$= H(p) + p\log 3$$

$$H(p) + p\log 3 \leqslant 1 \implies p_c \simeq 18.9\%$$