Noise in QC

0.) Reminder: decoherence

1) Reduced density matrix

2 Qubits systems :

$$|\Psi\rangle = a(t)|00\rangle + b(t)|01\rangle + c(t)|10\rangle + d(t)|11\rangle$$

$\rho = |\Psi\rangle\langle\Psi|$ ← a "pure" state

$$= \begin{pmatrix} |a|^2 & ab^* & ac^* & ad^* \\ a^*b & |b|^2 & bc^* & bd^* \\ a^*c & cb^* & |c|^2 & cd^* \\ a^*d & db^* & c^*d & |d|^2 \end{pmatrix}$$ in the local basis.

No access to second Qubit ⇒ what is theoretically accessible looking only at the first Qbit?

⇒ average of any observables involving the 1st Qubit is:

$$\langle O_1 \rangle = \mathrm{Tr}(\rho\, O_1 \otimes \mathbb{1}_2)$$
$$= \mathrm{Tr}_1(O_1 \mathrm{Tr}_2(\rho))$$

↑ partial trace

$\rho_1 \equiv \mathrm{Tr}_2(\rho)$

$\Rightarrow$ all we need to compute any $\langle O_1 \rangle$ is ρ_1

$$\mathrm{Tr}_2(A) = \sum_i \langle \psi_2^i | A | \psi_2^i \rangle$$

where $\{|\psi_2^i\rangle\}_i$ basis of the 2^{nd} Hilbert space only.

Here

$$\rho_1 = \langle 0_2 | \rho | 0_2 \rangle + \langle 1_2 | \rho | 1_2 \rangle$$
$$= |a|^2 |0\rangle\langle 0| + |b|^2 |0\rangle\langle 0| + ac^* |0\rangle\langle 1|$$
$$+ a^*c |1\rangle\langle 0| + bd^* |0\rangle\langle 1| + b^*d |1\rangle\langle 0|$$
$$+ |c|^2 |1\rangle\langle 1| + |d|^2 |1\rangle\langle 1|$$

$$= \begin{pmatrix} |a|^2 + |b|^2 & ac^* + bd^* \\ a^*c + b^*d & |c|^2 + |d|^2 \end{pmatrix}$$

coherence

Conservation of probability: $\mathrm{Tr}(\rho) = 1$

Purity: $\mathrm{Tr}(\rho^2) \leq 1$

$= 1$ iff pure state.

Taking $\rho_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

the "interference" terms

$$Tr(\rho_1^2) = A_{11}^2 + A_{12}A_{21} + A_{21}A_{12} + A_{22}^2$$

If coherence $= 0$ $\quad (A_{12} = A_{21} = 0)$

$$\Rightarrow Tr(\rho_1^2) = A_{11}^2 + A_{22}^2 \leq A_{11} + A_{22} \equiv 1$$

$=$ only if $A_{11} = 1$ & $A_{22} = 0$

or $A_{11} = 0$ & $A_{22} = 0$

$\Rightarrow$ in that case, the state is pure

ie: $|\psi\rangle = |0\rangle \otimes *$

or $|1\rangle \otimes *$

$\rightarrow$ not entangled

$\Rightarrow$ A non-separable state, when partial traced upon, becomes a mixed state.

2) Decoherence:

Bose - Einstein
Fermi - Dirac
Boltzmann

$$\rho_1 \xrightarrow{t \to \infty} \rho_{thermal} = \begin{pmatrix} f(T, E_0) & 0 \\ 0 & f(T, E_1) \end{pmatrix}$$

2 things happen:

coherence $\rightarrow 0$ (hence "decoherence")

$p_0, p_1 \rightarrow \underbrace{f(T, E_0), f(T, E_1)}$

give info about the temperature & energy $\neq$ between the two states, but not the initial qubit.

Typically:

* $p_0(t) \propto p_0(0)\, e^{-\frac{t}{T_1}} + f(T, E_0)\left(1 - e^{-\frac{t}{T_1}}\right)$

* $\text{coherence}(t) \propto \text{coherence}(0)\, e^{-\frac{t}{T_2}}$

$\Rightarrow$ How can this happen ?

2 ways:

- average over classical noise

- average over quantum noise (Lindblad)

coherence has an amplitude part that relaxes with the timescale T_1 and a phase part that <u>averages to zero</u> with the timescale T_{dephas}

$$\Rightarrow T_2 = f(T_1, T_{dephas})$$

I] Phenomenological noise : T_1, T_2 processes

1] Model

1. Spontaneous emission: there exists a process (whatever it is) that tends to have a $|+\rangle$ state spontaneously dexite to the $|-\rangle$ state by emitting a photon, changing the electro-mg field from $|0\rangle$ to $|\chi\rangle$ at the rate Γ.

2. Dephasing due to random changes in the frequency of the system (e.g. a cat playing with the magnetic field intensity button)

$$\omega(t) = \omega_0 + \delta\omega_j \quad \text{for } t \in [j\Delta t\,;(j+1)\Delta t[$$

with $\delta\omega_j$ taken at random.

Calling $\phi(t) = \int_0^t \omega(t)\,dt$, we have by hypothesis: $(t = n\Delta t)$

$$\langle e^{i\phi(t)} \rangle = e^{i\omega_0 t} \langle e^{i\delta\omega_1 \Delta t} e^{i\delta\omega_2 \Delta t} \cdots \rangle$$

$$= e^{i\omega_0 t} \langle e^{i\delta\omega\,\Delta t} \rangle^n$$

if $\delta\omega$ follows a gaussian distributi

$$= e^{i\omega_0 t} \left(e^{-\frac{(\sigma\omega\Delta t)^2}{2}}\right)^n$$

We have

$$\Rightarrow |\psi(t)\rangle = \alpha_0 e^{-i\frac{\phi_t}{2} - \frac{\Gamma}{2}t} |+\rangle \otimes |0\rangle$$
$$+ \alpha_0 e^{i\frac{\phi_t}{2}} \sqrt{1 - e^{-\Gamma t}} \, |-\rangle \otimes |\chi\rangle$$
$$+ \beta_0 e^{i\frac{\phi_t}{2}} |-\rangle \otimes |0\rangle$$

that describes well the system.

2) Extracting T_1 & T_2

We now compute $\langle \rho_{++}(t)\rangle$ & $\langle \rho_{+-}(t)\rangle$ to get T_1 and T_2

$$\downarrow$$

$$\langle \rho_{++}(t)\rangle \simeq e^{-\frac{t}{T_1}} \langle \rho_{++}(+\infty)\rangle$$

$$\langle \rho_{+-}(t)\rangle \simeq e^{-\frac{t}{T_2}} \langle \rho_{+-}(+\infty)\rangle$$

$$\langle \rho_{++}(t)\rangle = |\alpha_0|^2 e^{-\Gamma -}$$

$$\langle \rho_{--}(t)\rangle = |\alpha_0|^2 (1-e^{-\Gamma t}) + |\beta_0|^2$$

$$\langle \rho_{+-}(t)\rangle = \langle |\alpha_0|^2 e^{-i\phi(t)-\frac{\Gamma}{2}t}\sqrt{1-e^{-\Gamma t}} + \alpha_0\beta_0^* e^{-i\phi(t)-\frac{\Gamma}{2}t}\rangle$$

$$= |\alpha_0|^2 e^{-\frac{\Gamma}{2}t}\sqrt{1-e^{-\Gamma t}}\langle e^{-i\phi(t)}\rangle + \alpha_0\beta_0^* e^{-\frac{\Gamma}{2}t}\langle e^{-i\phi(t)}\rangle$$

$$\simeq e^{-\frac{\Gamma}{2}t}\, e^{-n\frac{(\Delta\omega\,\Delta t)^2}{2}}\, e^{\cdot i\omega_0 t} \times \underbrace{\left(|\alpha_0|^2\sqrt{1-e^{-\Gamma t}}+\alpha_0\beta_0^*\right)}_{\simeq\ \text{cst at long dime}}$$

$$\simeq e^{-i\omega_0 t-\left(\frac{\Gamma}{2}+\frac{\Delta\omega^2\Delta t}{2}\right)t}$$

$\rho_{+-}(\infty)$ at long time

$\rho_{+-}(0)$ at short time

$$\Rightarrow \frac{1}{T_1}=\Gamma \quad ; \quad \frac{1}{T_2}=\frac{\Gamma}{2}+\frac{\Delta\omega^2\Delta t}{2}=\frac{1}{2T_1}+\frac{1}{T_\phi}$$

$T_2 \leq 2T_1$

$\Rightarrow$ lose the phase before losing the level

$\Rightarrow$ phase is not a good way to encode information in general

III Accounting for calibration data

1] T_1 & T_2 per qubit

⇒ Same modelization as before
(+ do a bit better w/ uncertainty) * T_2 echo

In qiskit: (macro of phase_amplitude_damping_error
& phase_damping_error & amplitude_dampening)

```
error_gate = [thermal_relaxation_gate (t1, t2, gate_time)
              for t1, t2 in zip(T1s, T2s)]

noise_thermal = NoiseModel()
for j in range(#qubits)
    noise_thermal.add_quantum_error(error_gate[j],
                                    "gate", [j])

sim_thermal = AerSimulator(noise_model=noise_thermal)
```

→ how it is implemented? → see later.

2] Confusion matrices for measurements & reset (1 qubit)

Reminder: $\begin{pmatrix} p_0(\text{observed}) \\ p_1(\text{observed}) \end{pmatrix} = \begin{array}{c} \begin{pmatrix} P(0|0) & P(0|1) \\ P(1|0) & P(1|1) \end{pmatrix} \begin{array}{l} =1 \\ =1 \end{array} \end{array} \times \begin{pmatrix} p_0(\text{ideal}) \\ p_1(\text{ideal}) \end{pmatrix}$

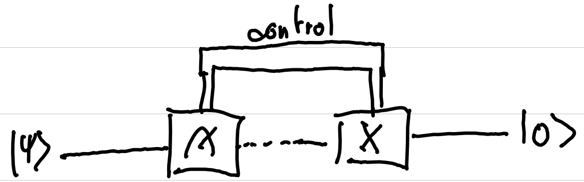
confusion matrix C

Mitigation: $\vec{p}(\text{ideal}) = C^{-1}\,\vec{p}(\text{observed})$

Reset, same principle. Typically C(reset) ~ C(meas)

⇒ cannot really tell them apart for mechanical reason.

Intuit:



In Qiskit:

Readout error [Confusion matrix of 1 qubit]

reset_error : same, but only proba.

3] Gate fidelity

$$\text{fidelity} = \left(\mathrm{Tr} \sqrt{\sqrt{\rho}\, \sigma \sqrt{\rho}} \right)^2 \leq 1 \qquad \begin{array}{l} \sigma : \text{target} \\ \rho : \text{observed} \end{array}$$

gate fidelity: σ target $= U_{\text{gate}}^{\text{perfect}}\, \sigma_{\text{perfect}}\, U_{\text{gate}}^{\text{perfect}\,-1}$

$$\rho = U_{\text{gate}}^{\text{flawed}}\, \sigma_{\text{perfect}}\, U_{\text{gate}}^{\text{flawed}\,-1}$$

$\Rightarrow$ strictly speaking, $\exists$ fidelity (target)
$\hookrightarrow$ important if non linear noise.

$\Rightarrow$ average over the targets:

$$\text{fidelity} = \frac{1}{\mathcal{N}} \int \sqrt{\ldots\ldots}\, d\sigma .$$

$$\Rightarrow \quad F = \int \left| \langle \Psi_{ini} | U_{\text{gate}}^{\text{perfect}\,-1}\, U_{\text{gate}}^{\text{flawed}} | \Psi_{ini} \rangle \right|^2 \times \frac{d\Psi_{ini}}{\mathcal{N}}$$

1 qubit $\downarrow$

$$\simeq \frac{1}{2} \left(\left| \langle 0 | \text{Gate}^{-1} | \Psi_{obs}(ini = 0) \rangle \right|^2 + \left| \langle 1 | \text{Gate}^{-1} | \Psi_{obs}(ini = 1) \rangle \right|^2 \right)$$

Ex: idle qubit

$-\boxed{I}-$ $\quad F = 0{,}95$

$\Rightarrow$ 95 % $-\boxed{I}-$

5 % $-\boxed{X}-$

$\Rightarrow$ compound, measured effect, so no direct Qiskit implementat°

(ex: T_1, T_2 proc affect fidelity)

→ simple implementat°: depolarisation noise

4) Depolarization noise

Phenomenological, "worst kind of noise"

1 parameter p:

$$\rho \xrightarrow{\text{depol}} (1-p)\,\rho + \frac{p}{\#\text{states}}\,\mathbb{1}$$

Fidelity of $\mathbb{1}$ with depol p:

$$\bar{F} = \frac{1}{\pi^2}\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi \; \langle \Psi | \rho_{\text{depol}} | \Psi \rangle$$

$$= (1-p) + \frac{p}{\#\text{states}}$$

$p = 0 \Rightarrow F = 1$

$p = 1 \Rightarrow F = \frac{1}{\#\text{states}}$ ← the fidelity of a random guess

In qiskit: depolarisation_error

5] Implementation / study of error

A] φ^{al} ground : Lindblad equation

$$i \partial_t |\Psi_{all}\rangle = H_{all} |\Psi_{all}\rangle$$

$$|\Psi_{all}\rangle \in \mathcal{H}_{Q.C.} \otimes \mathcal{H}_{env}$$

$$H_{all} = H_{QC} + H_{env} + H_{interaction}$$

$Tr_{env}(\,\cdot\,)$ + Markovian interaction only
+ good "reservoir"

$$\partial_t \rho_{Q.C.} = -\frac{i}{\hbar}[H_{Q.C.}, \rho_{Q.C.}]$$
$$+ \sum \gamma_i \left(K_i \rho K_i^+ - \frac{1}{2}\{K_i K_i^+, \rho\}\right)$$

$K_i \hookrightarrow$ Kraus operator

$\hookrightarrow$ if ρ is finite dimensional, $\exists$ a finite basis for the K_i ! (N^2-1)

Lindblad equation

B] Kraus operator

Noiseless: $|\psi\rangle \rightarrow U|\psi\rangle$

or equivalently

$$\rho \longrightarrow U \rho U^{\dagger}$$

With noise: not unitary:

$$\rho \xrightarrow{\text{gate}} U_{gate}\, \rho\, U_{gate}^{\dagger}$$

$\rho \xrightarrow{\text{w/ noise}} \sum \alpha_i\, K_i\, U_{gate}\, \rho\, U_{gate}^{\dagger}\, K_i^{\dagger}$

or $\sum \alpha_i\, U_{gate}\, K_i\, \rho\, K_i^{\dagger}\, U_{gate}^{\dagger}$

or in between.

There are alternative to Kraus formalism

IV Conclusions

- Noise "loses information", quantum & classical
- noisy state fully understood w/ density matrix
- coarsed grained understanding of noise through:
  - thermalization (T_1 & T_2)
  - reset & measurement confusions
  - fidelity
  - depolarization
- implementation in simulators through e.g. Kraus operations.
- Cross-talks.